PHASE NOISE IN OSCILLATORS: A MATHEMATICAL ANALYSIS OF LEESON'S MODEL

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Phase Noise in Oscillators: A Mathematical Analysis of Leeson's Model

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Abstract—**This paper is devoted to one aspect of the study of phase noise in oscillators: how an oscillator reacts to internal noise that occurs in the active element. The following theoretical analysis will lead us to express the Leeson's model in a more general form.**

Introduction

OSCILLATOR DESIGNERS and users often refer to the well-known Leeson's model [1] in order to explain the power-law spectral density of phase noise that occurs in real oscillators. This model is characterized by its author as being a "heuristic" model. More exactly, it is based on physical reasoning rather than on a detailed theoretical analysis. In this model, a feedback oscillator is viewed as a phase servo having positive feedback. Thus for modulation rates less than the half-bandwidth of the feedback ($\omega_0/2Q$, where ω_0 is the nominal angular frequency and Q is the resonator quality factor), phase error due to noise results in a frequency error determined by the phase–frequency relation. Outside the feedback bandwidth, the feedback has no effect. This reasoning leads to the following relationship:

$$S_\phi(\omega) = S_i(\omega)|L(j\omega)|^2$$

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where $S_\phi(\omega)$ is the power spectral density of phase noise at the output of the oscillator, $S_i(\omega)$ is the power spectral density of the internal phase uncertainties, and $|L(j(\omega)|^2 = [1 + (\alpha/\omega)^2]$ is Leeson's transfer function for an RLC resonator where $\alpha = \omega_0/2Q$.

The spectral density $S_i(\omega)$ is generally found to have a "flicker" component (ω^{-1} law) and a white-noise component (ω^0 law). For a high Q such as in a quartz oscillator, the expected laws for $S_\phi(\omega)$ are then ω^{-3} (flicker frequency noise), ω^{-1} (flicker phase noise), ω^0 (white-phase noise). This case where $\omega_0/2Q < \omega_c$ is shown graphically in Fig. 1.

This kind of spectral density has often been observed with quartz crystal oscillators [1], [2]. So, the validity of Leeson's model is enhanced by the good agreement between the predictions and experimental results. With this in mind, it seems advisable to establish a theoretical analysis verifying the validity of this model, and possibly expressing it in a more general form.

Mathematical Analysis

First, let us consider the block diagram of an oscillator (Fig. 2) which shows clearly the two basic elements: the amplifier and the selective feedback filter. It is worth noting that the general block diagram used for this mathematical analysis does not deal with the detail of the

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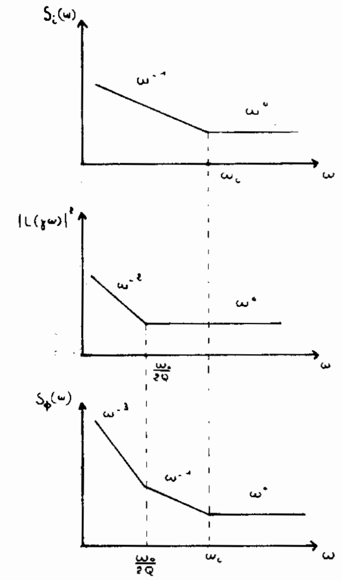

Fig. 1. Leeson's model for a high-Q oscillator ($\omega_c > \omega_0/2Q$). The scales are logarithmic.

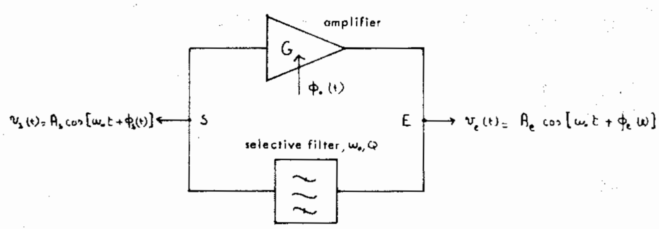

Fig. 2. Block diagram of an oscillator.

oscillator. Especially, the nonlinearities of the amplifier will not be considered.

Let us now suppose that the phase-noise modulation occurs in the oscillator active element as being $\phi_0(t)$; and we want to know how the oscillator reacts to this internal noise by expressing the power spectral density of phase noise at points E and S. According to the signal transmission theory, the bandpass filtering of a phase-modulated HF signal is identical to the low-pass filtering of the modulating signal in an equivalent filter. Thus the relationships which result from the null-phase condition upon the loop and from the filtering of $\phi_E(t)$ by the selective filter, may be written as

$$\phi_S(t) + \phi_0(t) = \phi_E(t)$$

$$\phi_S(t) = \int_{-\infty}^{+\infty} \phi_E(t) h_{BF}(t' - t) dt' = \phi_E(t) * h_{BF}(t)$$

where $h_{BF}(t)$ is the impulse response of the equivalent low-pass filter and $*$ denotes a convolution product. It can be demonstrated that the integral defining $\phi_S(t)$ converges for nearly all samples of $\phi_E(t)$ provided that the filter is linear and time invariant, and that the stationary random process $\phi_E(t)$ possesses a finite second-order moment [3].

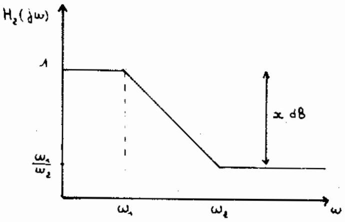

Fig. 3. Low-pass filter with finite attenuation.

By definition, the autocorrelation function of $\phi_0(t)$ may be written as

$$R_{\phi_0\phi_0}(t_1 - t_2) = \overline{\phi_0(t_1)\phi_0(t_2)}$$

(the bar denotes statistical average). We then have

$$R_{\phi_0\phi_0}(t_1 - t_2) = \overline{[\phi_E(t_1) - \phi_S(t_1)][\phi_E(t_2) - \phi_S(t_2)]}$$

$$R_{\phi_0\phi_0}(t_1 - t_2) = \overline{\phi_E(t_1)\phi_E(t_2)} + \overline{\phi_S(t_1)\phi_S(t_2)} - \overline{\phi_E(t_1)\phi_S(t_2)} - \overline{\phi_S(t_1)\phi_E(t_2)}.$$

In this relationship the autocorrelation and cross-correlation functions appear between phase fluctuations at the filter input and output. To develop this expression, consider relationships (1).

If $\phi_E(t)$ and $\phi_S(t)$ are two stationary, and jointly stationary of order two, random processes then:

$$R_{\phi_E\phi_E}(t_1 - t_2),\ R_{\phi_S\phi_S}(t_1 - t_2),\ R_{\phi_S\phi_E}(t_1 - t_2) \text{ and } R_{\phi_E\phi_S}(t_1 - t_2)$$

only depend on $t_1 - t_2 = \tau$. We have the following relations:

$$\begin{aligned} R_{\phi_S\phi_S}(\tau) &= R_{\phi_S\phi_E}(\tau) * h(\tau) \\ R_{\phi_E\phi_S}(\tau) &= R_{\phi_E\phi_E}(\tau) * h(\tau) \\ R_{\phi_E\phi_S}(\tau) &= R^*{}_{\phi_S\phi_E}(-\tau) \\ R^*{}_{\phi_S\phi_S}(-\tau) &= R_{\phi_S\phi_S}(\tau). \end{aligned} \tag{1}$$

Thus a relation may be established between $R_{\phi_0\phi_0}(\tau)$ and $R_{\phi_E\phi_E}(\tau)$ which reads

$$R_{\phi_0\phi_0}(\tau) = R_{\phi_E\phi_E}(\tau) + R_{\phi_S\phi_S}(\tau) - R_{\phi_S\phi_E}(\tau) - R_{\phi_E\phi_S}(\tau)$$

$$R_{\phi_0\phi_0}(\tau) = R_{\phi_E\phi_E}(\tau) + R_{\phi_E\phi_E}(\tau) * h^*(-\tau) * h(\tau) - R_{\phi_E\phi_E}(\tau) * h(\tau) - R_{\phi_E\phi_E}(\tau) * h^*(-\tau).$$

The Fourier transform of this expression gives the relationship between the power spectral density of internal phase uncertainties occurring in the amplifier and the power spectral density of phase noise at the output E of the oscillator

$$S_{\phi_0}(\omega) = S_{\phi_E}(\omega)[1 + H(j\omega)H^*(j\omega) - H(j\omega) - H^*(j\omega)]$$

hence

$$S_{\phi_E}(\omega) = S_{\phi_0}(\omega)\ [(H(j\omega) - 1)(H^*(j\omega) - 1)]^{-1}. \tag{2}$$

This expression is a more general form of Leeson's model, showing clearly the role of the equivalent low-pass filter

TABLE I

x(dB)	θ	10 Log θ
−20 dB	1.23	1 dB
−14 dB	1.56	2 dB
−10 dB	2.16	3.3 dB

TABLE II

x dB	−20 dB	−14 dB	−10 dB
y dB	−19 dB	−12 dB	−6.7 dB

transfer function. In the case of an *RLC* resonator, we obtain Leeson's transfer function as shown in the next section.

Applications

Two examples are treated in this section.

1) *RLC* filter denoting $H_1(j\omega)$ the equivalent low-pass transfer function of the *RLC* filter, then

$$H_1(j\omega) = \frac{\alpha}{j\omega + \alpha} \text{ with } \alpha = \frac{\omega_0}{2Q}$$

$$S_{\phi E}(\omega) = S_{\phi 0}(\omega) \left[\left(\frac{\alpha}{j\omega + \alpha} - 1 \right) \left(\frac{\alpha}{\alpha - j\omega} - 1 \right) \right]^{-1}$$

$$= S_{\phi 0}(\omega) \left(1 + \left(\frac{\alpha}{\omega} \right)^2 \right).$$

This is the result obtained in Leeson's original paper. If the signal is taken at the amplifier input (point S, see Fig. 1) we have:

$$S_{\phi S}(\omega) = S_{\phi 0}(\omega) |H_1(j\omega)|^2 = S_{\phi 0}(\omega) \left(\frac{\alpha}{\omega} \right)^2$$

2) Some filters have a finite attenuation outside the half-bandwidth (for example quartz-crystal resonator). As an example, let us now consider the following low-pass filter transfer function

$$H_2(j\omega) = \frac{\omega_1(j\omega + \omega_2)}{\omega_2(j\omega + \omega_1)}.$$

Substituting $H_2(j\omega)$ in (2) gives

$$S_{\phi E}(\omega) = S_{\phi 0}(\omega) \left\{ \left(\frac{\omega_1(j\omega + \omega_2)}{\omega_2(j\omega + \omega_1)} - 1 \right) \cdot \left(\frac{\omega_1(\omega_2 - j\omega)}{\omega_2(\omega_1 - j\omega)} - 1 \right) \right\}^{-1}$$

$$S_{\phi E}(\omega) = S_{\phi 0}(\omega) \frac{1}{(1 - \omega_1/\omega_2)^2} \left(1 + \frac{\omega_1}{\omega} \right)^2$$

$$= S_{\phi 0}(\omega) \theta \left(1 + \left(\frac{\omega_1}{\omega} \right)^2 \right).$$

This transfer function has the same behavior as Leeson's transfer function, except for a numerical constant θ, depending on the attenuation outside the filter bandwidth. In fact, for practical filters ($x \geqslant 20$ dB) the factor θ has negligible effect as shown in Table I.

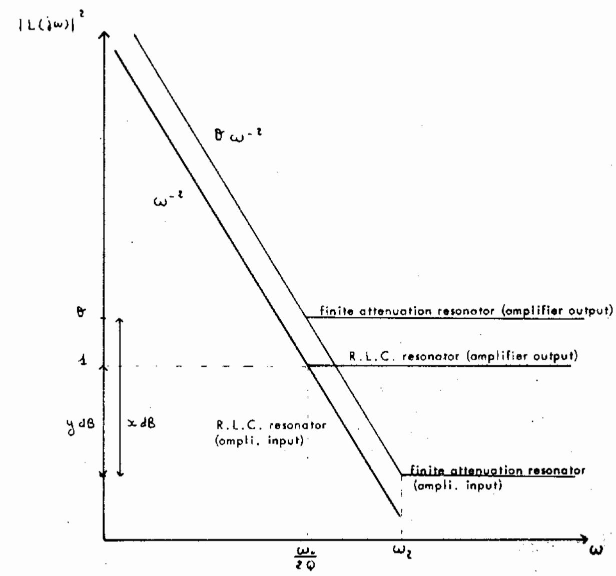

Fig. 4. Recap of transfer function.

If the signal is taken at the amplifier input, we have:

$$S_{\phi S}(\omega) = S_{\phi E}(\omega) |H_2(j\omega)|^2$$

$$= S_{\phi 0}(\omega) \frac{1}{(1 - \omega_1/\omega_2)^2} \left(1 + \left(\frac{\omega_2}{\omega} \right)^2 \right).$$

This transfer function is different from Leeson's model $[(\alpha/\omega)^2]$ in the way that the attenuation (y dB) outside the bandwidth ($\omega > \omega_2$) is close to x dB as shown in Table II. The various transfer functions are reported in Fig. 4.

Conclusion

It is hoped that this theoretical analysis of phase noise in oscillators may justify more completely the use of the power-law spectral density model [4]. Indeed, phase spectral densities of the form $k_\alpha f^{-\alpha}$ (with $\alpha = 3, 2, 1, 0$) appear not only as a useful picture but denote the behavior of phase noise in oscillators. This study must be considered as a step in oscillators noise theory and may lead to a better understanding of phase noise spectral densities in the case of complicated filter transfer functions.

Acknowledgment

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