

Characterization of Phase and Frequency Instabilities in Precision Frequency Sources: Fifteen Years of Progress

JACQUES RUTMAN, MEMBER, IEEE

Abstract—Precision frequency sources such as quartz oscillators, masers, and passive atomic frequency standards are affected by phase and frequency instabilities including both random and deterministic components. It is of prime importance to have a comprehensive characterization of these instabilities in order to be able to assess the potential utility of each source. For that purpose, many parameters have been proposed especially for dealing with random fluctuations. Some of them have been recommended by the IEEE Subcommittee on Frequency Stability and later by Study Group 7 on "Standard Frequencies and Time Signals" of the International Radio Consultative Committee (CCIR). Others are not so widely used but show interesting capabilities. This paper aims at giving a broad review of parameters proposed for phase and frequency instability characterization, including both classical widely used concepts and more recent less familiar approaches. Transfer functions that link frequency-domain and time-domain parameters are emphasized because they provide improved understanding of the properties of a given time-domain parameter or facilitate introducing of new parameters. As far as new approaches are concerned, an attempt has been made to demonstrate clearly their respective advantages. To this end, some developments that did not appear in the original references are presented here, e.g., the modified three sample variance $\Sigma_y^2(\tau)$, the expressions of $\langle \delta y_T^2 \rangle$, the interpretation of structure functions of phase and its relations with $\Sigma_y^2(\tau)$ and the Hadamard variance. The effects of polynomial phase and frequency drifts on various parameters have also been pointed out in parallel with those of random processes modeled by power-law spectral densities.

I. INTRODUCTION

PHASE AND FREQUENCY instability characterization has become of great concern to many engineers working in various fields since an increasing number of systems rely upon high-quality time and frequency sources such as quartz-crystal oscillators, frequency synthesizers, atomic frequency standards and clocks; also, the advent of frequency stabilized lasers has provided us with frequency standards in the optical range.

The following nonexhaustive list of systems illustrates the wide range of users [1], [2]:

- 1) Doppler radar systems with a narrow bandwidth receiver tuned to detect the shifted frequency return need high-performance transmitter oscillators and receiver local oscillators since any instability limits range resolution and sensitivity.
- 2) Oscillators are used in missiles and spacecrafts for a variety of purposes including guidance, tracking, and communi-

cations; frequency instabilities are harmful in all cases because they degrade system performance.

3) In range measurements where ranging signals are phase compared relative to a reference signal, instabilities in any of the oscillators involved introduce an uncertainty in the range estimate.

4) In communication systems, interference is reduced and performance is improved by better frequency control of the carrier frequencies. In digital communications, emphasis is put on the timing capability of the network clocks: an adequate performance measure, the maximum time interval error, is related to clocks phase and frequency instabilities. For long-term frequency fluctuations, a fractional value of 1 part in 10^{11} or better is recommended by the International Telegraph and Telephone Consultative Committee (CCITT) for the interconnection of several synchronous networks on the international level.

5) Of course, one should not forget the field of time and frequency metrology where sophisticated laboratory-type time and frequency standards are designed, constructed and operated, e.g., long cesium-beam and hydrogen devices. In fact, with few exceptions, most of the widely used stability measures have been developed by scientists working in the field.

A practical problem is that several groups of people with different backgrounds have had to find a common language for oscillator specifications.

The question was how to develop a useful and comprehensive characterization of phase and frequency instabilities that can be understood and applied by everyone. More specifically, in each field, everyone wanted to know how instabilities affect system performance and how the possible instability measures can be used to assess system performance. Of course, no single answer can be given and much work has been carried out during the last fifteen years to provide some useful answers.

Frequency stability was already recognized as an important problem at the beginning of the sixties: the special IEEE-NASA Symposium on Short-Term Frequency Stability held at Goddard Space Flight Center on November 23-24, 1964 [1] appeared as the first opportunity for cross-fertilization of ideas. Following this Symposium, an IEEE Subcommittee on Frequency Stability of the Technical Committee on Frequency and Time was created with the ultimate aim of providing a set of recommendations for standards and definitions on both short-term and long-term stability. A Special Issue of the PROCEEDINGS OF THE IEEE was devoted to frequency stability in February 1966 [2] to promote further exchange of information, the Subcommittee serving as Editorial Committee.

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The author is with the Laboratoire Primaire du Temps et des Fréquences, Bureau National de Métrologie, Observatoire de Paris, Paris 75014, France.

Several basic papers dealing explicitly with frequency stability characterization in both frequency and time domains and including the translations between them, were then published [3]-[7]. Of particular interest, the use of sample variances for time-domain characterization was developed in [7].

In May 1971, the Subcommittee issued a paper¹ [8] presented as technical background for an eventual IEEE standard definition (not yet adopted): two definitions of frequency stability were given together with translation relationships that play an important theoretical and practical role. Both have gained wide acceptance among manufacturers and users of precision frequency sources.

In the present paper, following a description of the mathematical models and basic definitions given in Section II, the parameters proposed as frequency stability measures will be studied in Section III (Fourier frequency domain) and Section IV (time domain).

In Section V, emphasis will be put on the role of the transfer functions which allow one to calculate the time-domain parameters (the variances) from the knowledge of the frequency-domain parameters (the spectral densities).

Up to now, we have mentioned only the basic concepts in wide use today namely the spectral densities of phase and frequency fluctuations and the two sample variance of averaged (fractional) frequency fluctuations. But in the meantime, other researchers have proposed new concepts leading to new time domain measures which are believed to exhibit some specific advantages relatively to the two-sample variance. These approaches will be presented in Sections VI-VIII with special emphasis on the relevant transfer functions relating these new parameters to the spectral densities. The links with more conventional concepts will be also outlined. It must be recognized that most of these new parameters are not widely used, possibly because their properties have not been pointed out clearly enough: our intent is not to recommend strongly the use of any new parameter but rather to demonstrate as clearly as possible their potential interest and usefulness. Readers are encouraged to try them whenever possible to test their utility.

From the experimental point of view, many test sets have been designed to measure frequency instability in both frequency and time domains. Although detailed technical descriptions are beyond the scope of this paper, some typical features of these systems will be recalled in Section IX.

To summarize, this paper aims at giving a broad review coverage of the published material on phase and frequency instability characterization, including both classical concepts and less familiar approaches. Since many points of view have been developed with great detail in the literature, only the main features can be covered here. An attempt has been made to present the material in a self-contained form that can be understood by nonspecialists. For more details, the reader is referred to the references² cited which have not been selected on the basis of being exhaustive but rather to point out the significant contributions.

II. MATHEMATICAL MODELS AND BASIC DEFINITIONS

Before dealing with mathematical models, let us define rather loosely that by frequency instability, we mean any unwanted frequency departure from a nominal value ν_0 . In

other words, frequency stability is the degree to which a source produces a constant frequency over a specified time interval. In practice, one often speaks of stability whereas applications are limited by instabilities; also, the measurement results which are much smaller than unity, e.g., 10^{-11} and so on, are indeed values of fractional (or relative or normalized) frequency instability.

The problem is hence the characterization of the unwanted frequency departures which are time-dependent because of the various physical mechanisms to be presented in this section.

A mathematical model is required for the oscillator quasi-sinusoidal output signal since the mere concept of frequency instability immediately implies that the signal is no longer a pure sinewave. Before establishing this model, it is useful to emphasize first the dichotomy between deterministic and random variations of the oscillator output frequency.

A. Deterministic Versus Random Frequency Variations.

Due to several physical mechanisms, the output frequency of any real source (even of the best quality) is continuously changing with time. Typical changes are as follows.

1) Systematic variations, also known as drifts or trends: they may be due to the aging of the resonator material (e.g., in quartz oscillators), but are also found in atomic frequency standards (e.g., some commercial cesium units exhibit a frequency drift of a few parts in 10^{13} per year). These extremely slow changes are often referred to as "long-term instability" and expressed in terms of parts in 10^x of frequency change per hour, day, month, or year, according to the device or the application. No statistical treatment is needed for the evaluation of these deterministic processes.

2) Deterministic periodic variations due to unwanted frequency modulation (FM) by periodic signals, such as the power-supply frequency and its harmonics. In some cases, quasi periodic frequency changes may arise from induced FM that may be traced to temperature, vibrations, pressure, etc. Of course, these environmental factors may induce more irregular frequency fluctuations.

3) Random fluctuations due to noise sources such as thermal, shot, and flicker noises encountered in electronic components. The related frequency fluctuations are often referred to as "short-term" instability since they become more and more significant when shorter time intervals are considered. Due to their random nature, statistical treatment is needed for their characterization.

4) It has been suggested that the frequency of an oscillator may suddenly take on a new and permanent average value. These frequency steps have limited documentation and it is not clear whether they are additional to the other parts of the model or just an unsuspected visual aspect of the data [10], [12]. Since no significant errors appear when these steps are ignored, they will not be introduced in the following models.

It is worth noting that expressions such as long-term or short-term have no absolute meaning; no objective limit can be given, valid for any oscillators or any applications. It is preferred to state explicitly the durations involved. The statistical parameters presented in the next sections are measures of instability due to random noises. However, in relation with the preceding discussion, some of their limitations must be pointed out.

1) Any frequency generator is influenced to some extent by its environment: since we cannot hope to give a unique prescription for handling every possible case, the proposed

¹Also published as NBS Technical Note 394, October 1970.

²The original notations have sometimes been changed to get a uniform terminology throughout this paper.

definitions of frequency instability will be independent of environmental factors. With proper design, most of these effects are secondary in importance in a laboratory environment, but in some applications (e.g., airborne) they may be by far the most important. Of course, high-quality oscillators are designed to minimize environmental sensitivities (e.g., magnetic shields in atomic frequency standards). In short, one should not expect an accurate prediction of frequency stability in a new environment when measurements have been made in another one [8].

2) A failure to separate systematic trends (drifts) from random fluctuations will, in most cases, affect any statistical measure in a very misleading way because of the dependency of statistical parameters on the number of samples. Whenever possible, systematics should be removed before statistical treatments. Linear frequency drifts can be rather easily subtracted but the elimination of trends is sometimes a tricky problem: then, it is wise to take off only the most obvious overall trend [9]. In any case, deterministic elements are not discarded: they are just recognized, evaluated, subtracted before statistical analysis and reintroduced in the final statement of oscillators specifications [10].

3) Measures of random frequency fluctuations may have a time dependence due to aging, e.g., frequency fluctuations in cesium clocks may increase after several years of continuous operation [9]. "Short term" measurements are hence valid only for a limited period.

4) As will be shown in Section IV, statistical measures are also affected by periodic frequency modulations.

Since we have an idea about the expected frequency departures, we are now in position to develop a model for the oscillator quasi-sinusoidal signal.

B. Model for the Oscillator Signal—Basic Definitions

The output signal of an ideal (noiseless, nondrifting) oscillator could be modeled as a pure sinusoid:

$$V(t) = V_0 \sin 2\pi\nu_0 t \quad (2.1)$$

where V_0 and ν_0 are the nominal amplitude and frequency, respectively, ($\omega_0 = 2\pi\nu_0$). For real oscillators, existing departures from V_0 and ν_0 have to be included in the model. The following general expression can be used:

$$V(t) = [V_0 + \epsilon(t)] \sin [2\pi\nu_0 t + \Phi(t)] \quad (2.2)$$

$\epsilon(t)$ is a random process³ denoting amplitude fluctuations around V_0 , also known as amplitude noise (see Section II-D). Before dealing with $\Phi(t)$, let us emphasize that any frequency variation immediately implies a related phase variation: more precisely, the instantaneous frequency is the time rate of change of phase divided by 2π . Therefore, $\Phi(t)$ denotes the phase modulation that may be traced to frequency departures introduced in Section II-A. A specific example reads as

$$\Phi(t) = D_1 t^2 + \Delta\Phi \sin 2\pi f_m t + \varphi(t) \quad (2.3)$$

³In many books on statistics, a real random process is denoted by: $\{z_k(t)\}$, $-\infty < t < +\infty$. It is an ensemble of real-valued functions which can be characterized through its probability structure. Each particular function $z_k(t)$ where k is fixed and t is variable is called a sample function: it may be thought of as the observed result of a single experiment. For the special class of ergodic random process, it is possible to derive statistical information about the entire random process from appropriate analysis of a single arbitrary sample function. For brevity, we will not include the symbol $\{ \}$ and the index k in the text. For an ergodic process, the index k is not needed since any one sample function is representative of all other sample functions.

where phase drift is modeled by a second-order polynomial (related to a linear frequency drift), periodic phase modulation is just a sinusoidal modulation and $\varphi(t)$ denotes a random process³ modeling the so-called phase noise associated with random frequency fluctuations.

In this paper, we are mainly interested in random instabilities. However, the other terms will be dealt with in Sections V and VIII since they may greatly affect the statistical measures.

Assuming negligible amplitude noise, the following simplified model may be used to study the random phase and frequency fluctuations [8]:

$$V(t) = V_0 \sin [2\pi\nu_0 t + \varphi(t)]. \quad (2.4)$$

By definition, the signal instantaneous frequency is

$$\nu(t) = \frac{1}{2\pi} \frac{d}{dt} (2\pi\nu_0 t + \varphi(t)) = \nu_0 + \frac{1}{2\pi} \frac{d\varphi(t)}{dt} \quad (2.5)$$

which may be rewritten as

$$\nu(t) = \nu_0 + \Delta\nu(t) \quad (2.6)$$

with

$$\Delta\nu(t) = \frac{1}{2\pi} \frac{d\varphi(t)}{dt} \quad (2.7)$$

where $\Delta\nu(t)$ is a random process modeling frequency noise. Of course, for high-quality oscillators:

$$|\Delta\nu(t)| \ll \nu_0 \quad (2.8)$$

for substantially all time t .

A useful parameter is the instantaneous fractional (or normalized) frequency deviation $y(t)$ defined as:

$$y(t) = \frac{\Delta\nu(t)}{\nu_0}. \quad (2.9)$$

The interest of such a dimensionless quantity is that it remains unchanged under frequency multiplications and divisions commonly encountered in systems (assuming noiseless multipliers and dividers). Also, it allows easier comparisons among sources having different nominal frequencies. It is sometimes of interest to introduce the parameter [8], [68]

$$x(t) = \frac{\varphi(t)}{2\pi\nu_0} \quad (2.10)$$

which is pulse expressed in units of time (sometimes referred to as phase time). Also, $x(t)$ is the instantaneous time error of a clock run from the oscillator having an instantaneous frequency $\nu(t)$. The following relation holds:

$$y(t) = \frac{dx(t)}{dt}. \quad (2.11)$$

C. Mathematical Difficulties (Model Pathologies)

The concepts of phase noise and frequency noise have just been introduced and modeled by random processes $\varphi(t)$ and $\Delta\nu(t)$, respectively. The theory of random processes is well developed⁴ and should be applied with as much mathematical rigor as possible. Related problems are discussed below but first, let us emphasize that models are used to represent the

⁴Many textbooks exist on the subject and should be consulted for more details.

physical world which is so complex that many details are ignored in the model: otherwise, the latter would become intractable. On the other hand, properties that have no direct meaningful counterparts in the real world have to be included in the model to make it tractable (stationarity of random processes is a well-known example). These problems have been discussed by Slepian [11] and by Barnes [10], [12] who gave a discussion of model pathologies for stability measurements.

1) *Stationarity of Phase Noise*: Stationarity of a random process is a precisely defined mathematical property⁴ meaning that statistical measures are time-independent. Clearly, lifetime limitations of physical experiments show that stationarity can only be a property of the model. The question is then whether and how one can use stationary models. Since we are looking for models that reasonably describe significant observables of real systems [8], and since stationary models are easy to use, stationarity must not be rejected from the lifetime argument.

For oscillator noise modeling, it is very convenient to assume the stationarity of $\varphi(t)$ since many theoretical results are valid only in this case, especially those related to correlation functions and spectral densities [5].

However, one should not misuse this assumption when dealing with phase and frequency fluctuations related by (2.7) which may be rewritten as [13]

$$\varphi(t) = \varphi_0 + \int_0^t 2\pi\Delta\nu(\theta) d\theta. \quad (2.12)$$

From this equation, and assuming that frequency noise is modeled by a stationary process $\Delta\nu(t)$ as can be justified by physical considerations, the random process $\varphi(t)$ is not generally stationary. In this case, one should not even use the correlation function $R_\varphi(\tau)$ of phase noise, a quantity that has been widely used in papers on frequency stability. As a particular case, white noise internal to the oscillator loop yields a so-called phase diffusion process analog to the mathematically ideal Brownian motion [12], [13] at least when one forgets bandlimiting effects existing in real oscillators.

To summarize, it is convenient to assume the stationarity of $\varphi(t)$ but one must check that this is not in conflict either with other parts of the model (e.g., properties of $\Delta\nu(t)$) or with some physical arguments.

2) *Existence of the Instantaneous Frequency*: A stationary random process can quite well have no derivative (it is differentiable in the mean-square sense if its correlation function has derivatives of order up to two). When this happens to phase noise, it means that the instantaneous frequency is not defined from the mathematical point of view. This may occur when discrete phase jumps are included in the model [13] but also with the ideal Brownian motion phase diffusion process [14]. In this case, one cannot even use the concept of instantaneous frequency defined by (2.5).

However, one has to realize that this is a model pathology, not a device pathology [12]. Bandwidth limiting that exists in real circuits yields fluctuations that can be modeled by processes where all orders of differentiation can be assumed to exist.

In conclusion, dealing with the instantaneous frequency, a key concept for frequency stability characterization, is valid only if it is properly defined mathematically.

3) *Nonstationary Models*: The limitations of the stationarity assumption have been pointed out, and one might well expect

that one would need nonstationary models for some problems. However, such models have seldom been found useful in describing oscillators frequency instability, except maybe those with stationary increments which belong to a particular class of nonstationary processes [15], [16]. In fact, some kind of nonstationarity has to be specified to get a tractable model.

Very often, flicker noise and other low-frequency (LF) divergent noises (defined in Section III-E) are associated with nonstationary properties but here again, experimental data and mathematical models should not be confused; a given set of data can often be modeled by either a stationary or a nonstationary model [12]. In terms of observation, neither model is more correct nor incorrect than the other: one usually prefers stationary models for simplicity (e.g., a stationary model has been developed for flicker frequency noise [17]).

Today, nonstationary random processes belong to a less familiar branch of statistics although much work has already been done in this field (see for example [18]). One has to be aware about these developments since they may prove useful for oscillator specification.

D. Amplitude Noise

In most treatments of frequency stability, the amplitude noise $\epsilon(t)$ appearing in (2.2) is neglected since:

- 1) it does not contribute directly to frequency instability although AM to PM conversion can occur in nonlinear devices;
- 2) most high-quality oscillators have some kind of amplitude stabilization: the degradation of spectral purity (defined in Section III-F) due to $\epsilon(t)$ usually is smaller than the one due to $\varphi(t)$. However, calculation shows that they have equal contributions when an external additive noise process [3], [19] is predominant;
- 3) limiter stages are used in many systems as an interface with a frequency standard, thus removing most AM noise.

To conclude, one has to worry about AM noise when the signal is processed by nonlinear devices or when interest is focused on signal spectral purity since AM noise degrades directly this performance (see (3.13)).

III. CHARACTERIZATION OF FREQUENCY STABILITY IN THE FOURIER FREQUENCY DOMAIN

In this section, we are dealing with the random fluctuations of phase, frequency and fractional frequency defined respectively in (2.4), (2.7), and (2.9). Since they are random quantities, statistical parameters are needed for their description, some of which are classically used in stationary random process theory, e.g., correlation functions and spectral densities.

Analysis in the Fourier frequency domain is of great importance both for theoretical purposes (in terms of included information) and for application purposes (in terms of power spreading in the frequency domain, a primary specification for many engineers). For these reasons, the spectral density concept has been widely used for oscillator stability characterization [3], [5], [8].

From here on, the word "frequency" will be used with two different meanings not to be confused: the time-dependent instantaneous frequency $\nu(t)$ of an oscillator, and the time-independent Fourier frequency, denoted by f , that will appear in spectral densities.

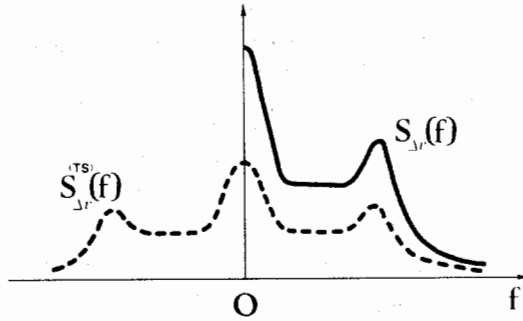


Fig. 1. Two-sided and one-sided spectral densities of frequency fluctuations.

A. Spectral Density of (Fractional) Frequency Noise

Assuming that the frequency noise is modeled by a stationary process $\Delta\nu(t)$, the spectral density $S_{\Delta\nu}(f)$ is defined as the Fourier transform of the correlation function [5]

$$R_{\Delta\nu}(\tau) = \langle \Delta\nu(t) \Delta\nu(t - \tau) \rangle \quad (3.1)$$

where $\langle \rangle$ denotes the average of the quantity inside the brackets. More precisely, this approach yields the two-sided (TS) spectral density defined for $-\infty < f < +\infty$; it is a real non-negative and an even function of f (Fig. 1):

$$S_{\Delta\nu}^{(TS)}(f) = \int_{-\infty}^{+\infty} R_{\Delta\nu}(\tau) \exp(-i2\pi f\tau) d\tau \quad (3.2)$$

For theoretical considerations, TS spectral densities often simplify the calculations. For experimental purposes, the one-sided spectral density $S_{\Delta\nu}(f)$ defined as follows is used:

$$S_{\Delta\nu}(f) = 2S_{\Delta\nu}^{(TS)}(f), \quad \text{for } 0 \leq f < \infty, \quad \text{otherwise zero.} \quad (3.3)$$

In practice, this quantity may be measured for example by filtering, squaring, and averaging operations on sample records of $\Delta\nu(t)$ or by digital techniques involving the fast Fourier transform (FFT). Of course, as for any statistical parameter, only useful estimates may be obtained experimentally [20].

The following equation links the mean-square value of $\Delta\nu(t)$ to the area under $S_{\Delta\nu}(f)$:

$$\langle \Delta\nu^2(t) \rangle = \int_0^{\infty} S_{\Delta\nu}(f) df. \quad (3.4)$$

The dimensions of $S_{\Delta\nu}(f)$ are Hz^2/Hz , i.e., Hz .

In the same way, the spectral density $S_y(f)$ of $y(t)$ may be introduced: it has the dimensions of $1/\text{Hz}$.

The IEEE Subcommittee has proposed to use $S_y(f)$ as a definition for the measure of frequency stability in the Fourier frequency domain: oscillators at different nominal frequencies may thus be meaningfully compared by plotting their $S_y(f)$ on the same figure.

It must be recognized that correlation functions and spectral densities carry exactly the same information about the random process: however, the spectral density format is very often more desirable for engineering purposes.

B. Spectral Density of Phase Noise

Despite the problems that may arise concerning the stationarity of phase noise, it is very convenient to assume that $\varphi(t)$ is a stationary process: its one-sided spectral density $S_\varphi(f)$ is

then defined from $R_\varphi(\tau)$ and has the dimensions of $\text{radian}^2/\text{Hz}$.

The concept of $S_\varphi(f)$ is widely used for several reasons:

- 1) it is measured with the well-known phase detector technique implemented in many laboratories (see Section IX);
- 2) it is very simply related to $S_y(f)$, the recommended measure (3.7);
- 3) under certain assumptions, it provides an estimate of signal spectral purity whose direct measurement is often difficult for high-quality sources (see Section III-F).

C. Translations among Frequency-Domain Measures

From (2.9):

$$S_y(f) = \frac{1}{\nu_0^2} S_{\Delta\nu}(f). \quad (3.5)$$

Since $2\pi\Delta\nu(t)$ is the time derivative of $\varphi(t)$ (2.7), the following relationship holds in the frequency domain:

$$S_{\Delta\nu}(f) = f^2 S_\varphi(f). \quad (3.6)$$

Combining (3.5) and (3.6) yields

$$S_y(f) = \left(\frac{f}{\nu_0}\right)^2 S_\varphi(f). \quad (3.7)$$

Thus these three spectral densities carry essentially the same information, each of them being best suited for a given class of applications.

Also, the spectral density $S_x(f)$ of time error $x(t)$ defined in (2.10) may be introduced and:

$$S_y(f) = 4\pi^2 f^2 S_x(f). \quad (3.8)$$

Last, a general remark is that the modulating processes $\varphi(t)$ and $\Delta\nu(t)$ are slow relatively to the carrier $\sin 2\pi\nu_0 t$: therefore, their spectral densities take significant values only for f much smaller than ν_0 .

D. Estimation of Spectral Densities

Spectral densities are theoretical concepts involving infinite duration processes, infinite frequency range and true averages. In practice, only finite-duration processes are available and spectrum analyzers have nonzero-bandwidth frequency windows, limited dynamic ranges, slewing rates (for swept models), lower and upper Fourier frequency limits, etc.

As a consequence, experimental knowledge of spectral densities suffers from several kinds of limitations.

1) The Fourier frequency range is limited: a lower limit of about $f = 1$ Hz is reached by several LF analog spectrum analyzers and about 10^{-3} Hz may be reached with digital techniques using the FFT. An upper limit of tens of kHz is typical for LF analyzers.

2) Statistical errors result from the use of a finite sample of observations [20]: error bars should be specified with any experimental result. Other errors are due to data acquisition and processing techniques: systematic errors greater than 10 dB can occur, a few dB appearing as a typical value. An accuracy of 0.2 dB can be obtained only by exercising considerable caution including the study of the analyzer's filter shape effects [21].

3) Some analyzers yield only the spectral densities for discrete values of f : narrow lines due to periodic modulation may be overlooked with such instruments.

Nevertheless, the spectral density concept plays a key role in stability characterization since it provides an unambiguous identification of the noise processes encountered in real oscillators as shown in the next paragraph.

E. The Power-Law Spectral Density Model

From spectral density measurements made in many laboratories on various sources including quartz-crystal oscillators, masers, passive atomic standards, and other microwave oscillators, it appears that experimental results may quite well be modeled by power law curves. For $S_y(f)$, the following model has been found useful [8]:

$$S_y(f) = h_\alpha f^\alpha. \quad (3.9)$$

The exponent α typically takes the integral values $-2, -1, 0, +1, +2$ and is a characteristic of the kind of noise. The constant h_α is a measure of the noise level. Nonintegral values of α may also be considered.

For a given oscillator, $S_y(f)$ is the sum of two or three such terms, the others being negligible.

1) *Classification of Power Laws*: The integral power laws may be designated using the classical terminology of "white noise" for a noise whose spectral density is a constant (independent of f), "flicker noise" when it varies as f^{-1} and "random walk" when it varies as f^{-2} . Table I summarizes these laws.

Note that the quantity of interest (frequency versus phase) must be explicitly stated since both might exist simultaneously. Experimental data are usually plotted on log-log scales where power laws appear as straight lines: the slopes are then easily recognized (if measurement accuracy is not too bad) and hence the kinds of noise present in the oscillator.

2) *Physical Origins*: White and flicker electrical noise sources are ever present in electronic components. In oscillators, depending on their location in the circuits, their additive contribution to the signal yields either phase modulation (for external noises) or frequency modulation (for internal noise) by the original noise processes [3]–[5]. Moreover, the so-called "multiplicative" noises directly modulate the phase or the frequency according to the physical mechanisms considered.

More specific statements can be made [22].

a) Random walk frequency noise usually relates to the oscillator environment (temperature, vibration, shocks, etc.).

b) Flicker frequency noise sources are not yet fully understood but are thought to be related to the resonator in quartz oscillators and to electronics and environment in atomic frequency standards [23].

c) White frequency noise arises from additive white noise sources internal to the oscillator loop, such as thermal noise [3]. It is also found in passive atomic standards where white noise sources directly modulate the locked oscillator output frequency via the control element [24].

d) Flicker phase noise is usually added by noisy electronics, e.g., by output amplifiers or frequency multipliers, and can be reduced by radio frequency (RF) negative feedback and component selection [25].

e) White phase noise is usually due to additive white noise sources external to the oscillator loop [3]. Bandpass filtering the oscillator output signal is then useful.

3) *Power-Law Pathologies* [12]: Power laws are useful models for data obtained over a limited Fourier frequency interval. Extrapolation over the whole frequency domain from

TABLE I
THE POWER-LAW MODEL FOR SPECTRAL DENSITIES

$S_y(f)$	$S_\varphi(f)$	Designation
$h_2 f^{-2}$	$\nu_0^2 h_2 f^{-4}$	Random walk frequency noise
$h_1 f^{-1}$	$\nu_0^2 h_1 f^{-3}$	Flicker frequency noise
h_0	$\nu_0^2 h_0 f^{-2}$	White frequency noise
$h_1 f$	$\nu_0^2 h_1 f^{-1}$	Flicker phase noise
$h_2 f^2$	$\nu_0^2 h_2$	White phase noise

$f = 0$ to $f = \infty$, i.e., beyond the range of applicability given by observations, yields mathematical difficulties such as infinite power and divergence of some time-domain stability measures. But this results from the use of an unrealistic model ignoring the constraints of the real world such as finite bandwidths and finite duration which prevent both high-frequency (HF) and LF divergence from being observable. Also, the instantaneous frequency generated by any real source is bounded and thus, models of $S_y(f)$ with infinite area are physically unrealistic.

In practice, the power-law model is often used as follows [8]:

$$S_y(f) = \begin{cases} \sum_{\alpha=-2}^{+2} h_\alpha f^\alpha, & 0 \leq f \leq f_h \\ 0, & f > f_h \end{cases} \quad (3.10)$$

where a sharp upper cutoff frequency f_h is introduced. Sometimes, the actual shape of the cutoff is of importance [6] and must then be specified. As will be seen in the next section, time domain stability measures sometimes depend on f_h which must then be given with any numerical result, although no recommendation has been made for its value (1 to 10 kHz is typical).

A lower cutoff frequency is usually not included in the power-law model since useful time-domain measures have been defined such that convergence arises for $S_y(f)$ given by (3.10). Moreover, no experimental results indicate clearly the value that should be included: measurements have just shown that flicker-type spectral densities extend down to very-low Fourier frequencies, e.g., smaller than one cycle per month as observed on quartz oscillators [26]. No flattening has yet been observed.

F. Oscillator Spectral Purity

When dealing with the frequency domain for oscillator specification, it is of value to introduce the concept of spectral purity, i.e., a frequency-domain estimate of the quality of the oscillator quasi-sinusoidal output signal $V(t)$.

To this end, the spectral density $S_\varphi(f)$ of the complete signal $V(t)$ is used and often referred to as RF spectrum⁵ since it

⁵Strictly speaking, ν_0 is not restricted to the RF range and may be in the microwave, infrared, or even visible according to the source under test.

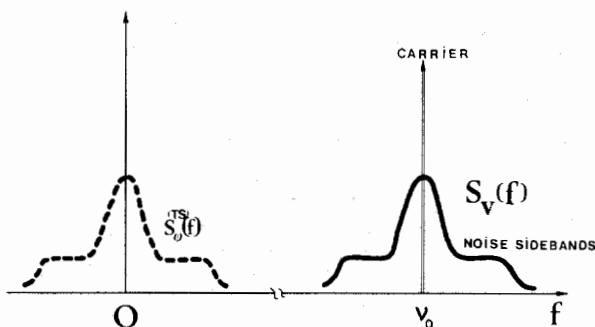


Fig. 2. Signal spectral purity: The RF spectrum.

takes significant values only around ν_0 . Knowledge of it is of prime importance in several applications such as radar, spectroscopy, frequency synthesis and communications.

Nevertheless, $S_v(f)$ is not considered as being a good primary measure of frequency stability since amplitude fluctuations $\epsilon(t)$ also contribute to $S_v(f)$ [8]. Moreover, it is not simply related with other measures of frequency stability in the most general cases.

It is only when AM noise is negligible and the root-mean-square (rms) value of $\varphi(t)$ is much smaller than one radian that a simple approximate relationship may be derived [3], [5]:

$$S_v(f) \approx \frac{V_0^2}{2} \{ \delta(f - \nu_0) + S_\varphi^{(TS)}(f - \nu_0) \}. \quad (3.11)$$

As pictured on Fig. 2, the continuous noise sidebands around the discrete carrier centered at ν_0 have then the shape of the spectral density of phase noise. This so-called low-modulation-index approximation may be used for high-quality frequency standards for which the above assumptions are reasonably fulfilled. Sometimes, the complete formula is needed, e.g., to forecast spectral purity of a signal after very high ratio frequency multiplication [27] as realized for laser-frequency measurements.

Neglecting AM noise, the complete relationship between phase noise spectral density and RF spectrum⁶ reads as [28]

$$S_v(f) = \frac{V_0^2}{2} e^{-\langle \varphi^2 \rangle} \left\{ \delta(f - \nu_0) + S_\varphi^{(TS)}(f - \nu_0) + \sum_{n=2}^{\infty} \frac{1}{n!} [S_\varphi^{(TS)}(f) \otimes^{n-1} S_\varphi^{(TS)}(f)]_{\nu_0} \right\}. \quad (3.12)$$

In this equation, $\langle \varphi^2 \rangle$ denotes the mean-square value of $\varphi(t)$ which is assumed to be a stationary Gaussian random process. $[S_\varphi^{(TS)}(f) \otimes^{n-1} S_\varphi^{(TS)}(f)]_{\nu_0}$ denotes $n-1$ convolutions of $S_\varphi(f)$ with itself, followed by a translation around the carrier frequency ν_0 . Due to the infinite sum, the noise sidebands no longer have the shape of $S_\varphi(f)$.

More precisely, an increase of $\langle \varphi^2 \rangle$ yields:

- 1) a decrease of carrier power (due to the exponential term);
- 2) a change of the noise sidebands shape (due to the increasing contribution of the convolutions);
- 3) a decrease of carrier-to-noise power ratio since the total area under $S_v(f)$ remains constant ($V_0^2/2$).

⁶RF spectra of signals randomly modulated in amplitude, phase and frequency have been studied by several authors with other backgrounds than frequency standards specification, see for example Middleton [29].

If necessary, the contribution of AM noise may be taken into account as shown in the following equation:

$$S_v(f) \approx \frac{V_0^2}{2} \{ \delta(f - \nu_0) + S_\varphi^{(TS)}(f - \nu_0) + S_A^{(TS)}(f - \nu_0) \} \quad (3.13)$$

where $S_A^{(TS)}(f)$ is the two-sided spectral density of the fractional amplitude fluctuations $A(t) = \epsilon(t)/V_0$.

Direct measurements of high-quality oscillator's spectral purity is very difficult because of the ultralow sideband levels encountered and the limited capabilities of even the best RF spectrum analyzers. The measurement of $S_\varphi(f)$ is often used as a means to estimate $S_v(f)$ through (3.11).

IV. CHARACTERIZATION OF FREQUENCY STABILITY IN THE TIME DOMAIN

Since instabilities in oscillators are time variations of the quantities of interest (phase, frequency, fractional frequency), they may be characterized by a measure of the variations that occur over a specified time interval τ : this is the basis of the so-called time-domain characterization (time-interval or averaging-time domain would be more appropriate). Since random phenomena are involved, the relevant measures are again given in terms of statistical parameters. These values are usually plotted versus τ which may vary from say milliseconds to days, months, or years.

Some important applications of frequency standards directly rely on the time domain behavior of the output signal, e.g., timekeeping [30]. In this case, a time-domain instability measure is really what is needed, rather than a frequency-domain one. The techniques described below are not restricted to power-law spectral densities and may be used with other noise types.

A major difficulty will arise from the fact that many parameters may indeed be considered as time-domain measures: one gives here a presentation showing their respective properties and relationships.

A. Basic Measurements

The oscillator instantaneous frequency $\nu(t)$ defined in (2.5) is not an observable since any frequency-measurement technique does involve a finite time interval over which the measurement is performed; for example, a digital frequency counter counts the number of cycles n_k of the input signal during the time interval τ beginning at t_k provided by its time base driven by a reference oscillator.

Therefore, the average value of $\nu(t)$ over a time interval τ beginning at t_k provides a more useful quantity directly related to an experimental result:

$$\langle \nu(t) \rangle_{t_k, \tau} = \nu_0 + \frac{1}{\tau} \int_{t_k}^{t_k + \tau} \Delta \nu(\theta) d\theta = \frac{n_k}{\tau}. \quad (4.1)$$

The normalized quantity \bar{y}_k defined as follows is widely used [8]:

$$\bar{y}_k = \frac{1}{\tau} \int_{t_k}^{t_k + \tau} y(\theta) d\theta \quad (4.2)$$

and then

$$\langle \nu(t) \rangle_{t_k, \tau} = \nu_0 (1 + \bar{y}_k) = \frac{n_k}{\tau}. \quad (4.3)$$

From (2.7) and (2.9), it can be demonstrated that

$$\bar{y}_k = \frac{\varphi(t_k + \tau) - \varphi(t_k)}{2\pi\nu_0\tau} \quad (4.4)$$

where the numerator denotes the phase error accumulated from t_k to $t_k + \tau$, also known as the first phase increment or first difference of the phase [30], [31].

Since \bar{y}_k is easily related to experimental results given by counting techniques, it will be used in the following to define the time-domain parameters. More precisely, one individual measurement of duration τ provides one sample \bar{y}_k ; repeated measurements of \bar{y}_k are necessary for a statistical treatment yielding a meaningful measure of instability over τ .

B. The True Variance [3]-[6]

Due to the random fluctuations of $y(t)$ in real oscillators, repeated measurements of \bar{y}_k give numerical values that are randomly scattered i.e., they are samples of a random variable: for each value of τ , a statistical characteristic of the dispersion of the \bar{y}_k provides a time-domain measure of instability over τ .

To this end, the variance σ^2 (or the standard-deviation σ) is widely used in statistics. Assuming that $y(t)$ and hence the \bar{y}_k have a zero mean, the variance is equal to the mean square value of \bar{y}_k :

$$\sigma^2[\bar{y}_k] = \langle \bar{y}_k^2 \rangle. \quad (4.5)$$

The bracket $\langle \rangle$ denotes either a statistical average calculated over an infinite number of samples at a given instant t_k , or an infinite time average made over one sample of $y(t)$ (ergodicity of $y(t)$ is assumed). Because of the infinite number of samples or infinite duration implied in its definition, this variance is an idealization often referred to as the true variance. It will be denoted as $I^2(\tau)$ to indicate that it is a measure of instability over a time interval τ .

For stationary frequency fluctuations, the general shape of $I(\tau)$ reported on Fig. 3 shows the following limits:

$$\lim_{\tau \rightarrow 0} I(\tau) = \sqrt{\langle y^2(t) \rangle} \quad (4.6)$$

$$\lim_{\tau \rightarrow \infty} I(\tau) = 0. \quad (4.7)$$

The first limit corresponds to ideal instantaneous frequency measurements ($\tau = 0$) and hence is equal to the rms value (i.e., the standard deviation) of $y(t)$. The upper limit means that stationary fluctuations would be completely averaged out for $\tau = \infty$ and hence the dispersion of the results would be zero. Of course, none of these limits are observables. The decrease of $I(\tau)$ as τ increases indicates better averaging of stationary random fluctuations.

Properties but also limitations⁷ of the usefulness of $I(\tau)$ will be studied in Section V through its relationship to $S_y(f)$.

C. The Sample Variance [7], [8]

The true variance $I^2(\tau)$ is a theoretical idealization since it involves an infinite number of data: practical estimates of it must be based upon a finite number of samples \bar{y}_k . The so-called sample variance is defined from an ensemble of N samples \bar{y}_k with $k = 1, 2, 3, \dots, N$ and $t_{k+1} = t_k + T$. The

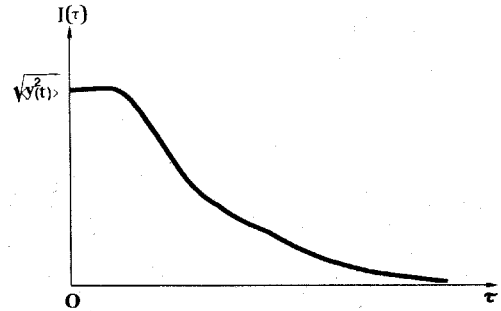


Fig. 3. Instability in the time domain: Square-root of the true variance for stationary frequency noise.

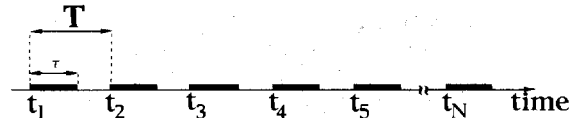


Fig. 4. The measurement cycle in the time domain.

corresponding measurement cycle is shown on Fig. 4, where T is the repetition interval for individual measurements of duration τ ($T = \tau +$ dead time between measurements) and t_1 is arbitrary.

Several related definitions may be given for the sample variance [32].

1) *First Definition:* Following the general definition of the variance of a random variable x , namely $\sigma^2(x) = \langle (x - \langle x \rangle)^2 \rangle$, a logical definition of the sample variance reads as

$$\sigma_{(1)}^2(N, T, \tau) = \frac{1}{N} \sum_{i=1}^N \left(\bar{y}_i - \frac{1}{N} \sum_{j=1}^N \bar{y}_j \right)^2. \quad (4.8)$$

This quantity is itself a random variable, N being the sample size. Several factors may be used to characterize the "goodness" of an estimator [20], bias being one of them: an estimator is unbiased if its mean value equals the true parameter, namely if

$$\langle \sigma_{(1)}^2(N, T, \tau) \rangle = I^2(\tau). \quad (4.9)$$

This equality should hold for any value of N , T , and τ and any kind of noise. Thus the relation between $\langle \sigma_{(1)}^2(N, T, \tau) \rangle$ and $I^2(\tau)$ has to be studied carefully: let us consider the specific case of great importance where the samples are adjacent ($T = \tau$); then, the following relation holds [7], [32]:

$$\langle \sigma_{(1)}^2(N, \tau, \tau) \rangle = I^2(\tau) - I^2(N\tau). \quad (4.10)$$

For stationary frequency noise (Fig. 5):

$$\lim_{N \rightarrow \infty} \langle \sigma_{(1)}^2(N, \tau, \tau) \rangle = I^2(\tau). \quad (4.11)$$

Hence, for finite N , the above estimator is biased for any kind of stationary frequency noise and the bias may be calculated from the theoretical knowledge of $I^2(N\tau)$. The other definitions of sample variance given below are such that the bias is indeed zero for a given definition and a corresponding kind of noise.

2) *Second Definition and White Frequency Noise:* For white frequency noise (see Table I), equation (5.6) yields $I^2(\tau) = h_0/2\tau$ and hence,

$$\langle \sigma_{(1)}^2(N, \tau, \tau) \rangle = \left(1 - \frac{1}{N} \right) I^2(\tau). \quad (4.12)$$

⁷ Especially, $I^2(\tau)$ does not converge for some kinds of noise such as flicker frequency noise.

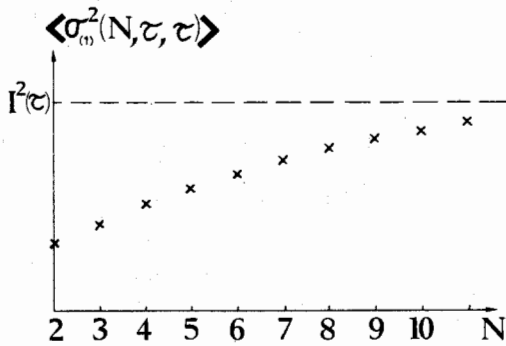


Fig. 5. The sample variance versus N for stationary frequency noise (first definition).

In this case, the following definition provides an unbiased estimator:

$$\sigma_{(2)}^2(N, T, \tau) = \frac{1}{N-1} \sum_{i=1}^N \left(\bar{y}_i - \frac{1}{N} \sum_{j=1}^N \bar{y}_j \right)^2. \quad (4.13)$$

For $T = \tau$ and white frequency noise

$$\langle \sigma_{(2)}^2(N, \tau, \tau) \rangle = I^2(\tau). \quad (4.14)$$

The factor $1 - 1/N$ appearing in (4.12) is found also for other stationary noises when T is much greater than the correlation time of $y(t)$, i.e., when the samples \bar{y}_k are independent: this is a classical result in statistics [20].

3) *Third Definition and White Phase Noise* [32]: For white phase noise including a cutoff frequency f_h , equation (5.6) yields $I^2(\tau) = h_2 f_h / 2\pi^2 \tau^2$ for $2\pi f_h \tau \gg 1$; then,

$$\langle \sigma_{(1)}^2(N, \tau, \tau) \rangle = \left(1 - \frac{1}{N^2} \right) I^2(\tau). \quad (4.15)$$

In this case, the following sample variance is unbiased for any $N \geq 2$ and $T = \tau$:

$$\sigma_{(3)}^2(N, T, \tau) = \frac{N}{N^2 - 1} \sum_{i=1}^N \left(\bar{y}_i - \frac{1}{N} \sum_{j=1}^N \bar{y}_j \right)^2. \quad (4.16)$$

To summarize, we are faced with several definitions of the sample variance, many values of $N \geq 2$, and a wide range of possible values of $T \geq \tau$: also, the value of f_h is sometimes of prime importance (e.g., for white and flicker phase noises).

To achieve the goal of recommending a unique time-domain measure, some choices must be made.

D. The Zero-Dead-Time Two-Sample Variance

To allow meaningful comparisons of measurements in the time domain, one sample variance must be chosen with specified values of N , T , and $f_h(\tau)$ being the independent variable.

Following the work of Allan [7], the IEEE Subcommittee on Frequency Stability has proposed the following time-domain measure [8]:

- 1) the average value of $\sigma_{(2)}^2(N, T, \tau)$ given by (4.13) is used, namely $\langle \sigma_{(2)}^2(N, T, \tau) \rangle$ often denoted as $\langle \sigma_y^2(N, T, \tau) \rangle$;
- 2) the sample size is taken as $N = 2$;
- 3) adjacent samples (zero-dead-time) are assumed, and hence $T = \tau$.

The resulting measure reads as

$$\langle \sigma_y^2(2, \tau, \tau) \rangle = \left\langle \sum_{i=1}^2 \left(\bar{y}_i - \frac{1}{2} \sum_{j=1}^2 \bar{y}_j \right)^2 \right\rangle. \quad (4.17)$$

The short notation $\sigma_y^2(\tau)$ is widely used for that quantity

$$\sigma_y^2(\tau) = \frac{1}{2} \langle (\bar{y}_2 - \bar{y}_1)^2 \rangle. \quad (4.18)$$

It is also known as the Allan variance or the pair variance.

Several comments are of value.

1) Since infinite duration is implied in the average denoted as $\langle \rangle$, $\sigma_y^2(\tau)$ is a theoretical measure in the same sense as $I^2(\tau)$: however, as shown in Section V, it has a greater practical utility since it exists for the five power laws encountered in real oscillators (Table I). Also, simple experimental estimators may be devised for $\sigma_y^2(\tau)$ (see Section IV-E).

2) From the discussion on the various sample variances, it appears that $\sigma_y^2(\tau)$ is biased except for white frequency noise, but one has to remember that other noises are found as well in oscillators; from (4.10) and (4.17), one gets:

$$\sigma_y^2(\tau) = 2[I^2(\tau) - I^2(2\tau)]. \quad (4.19)$$

For white frequency noise and white phase noise, the following relations are valid, respectively,

$$\sigma_y^2(\tau) = I^2(\tau) \quad (\text{no bias}) \quad (4.20)$$

and

$$\sigma_y^2(\tau) = \frac{3}{2} I^2(\tau) \quad 2\pi f_h \tau \gg 1 \quad (\text{bias}). \quad (4.21)$$

3) The choice of $N = 2$ in the preferred definition yields an easy measurement process since only pairs of samples are involved: this choice is really the key feature in the definition of $\sigma_y^2(\tau)$.

4) Although no recommendation was made for the value of f_h , it has to be specified with any experimental results.

E. Estimates of the Two-Sample Variance

Only estimates of $\sigma_y^2(\tau)$ can be obtained from a finite number of samples \bar{y}_k and an inherent statistical uncertainty exists. With m values of \bar{y}_k , a possible estimator reads as [8]

$$\hat{\sigma}_y^2(\tau, m) = \frac{1}{2(m-1)} \sum_{i=1}^{m-1} (\bar{y}_{i+1} - \bar{y}_i)^2. \quad (4.22)$$

This quantity is itself a random variable whose variance (i.e., the variance of the variance) may be used to calculate the error bars on the plot of $\sigma_y(\tau)$ versus τ . For Gaussian noises, Lesage and Audoin [33] have shown that the error bars (confidence intervals) for power laws are given by

$$E_\alpha \approx \sigma_y(\tau) K_\alpha m^{-1/2}, \quad \text{for } m > 10 \quad (4.23)$$

with $K_2 = K_1 = 0.99$, $K_0 = 0.87$, $K_{-1} = 0.77$, $K_{-2} = 0.75$.

Also, other estimators of $\sigma_y^2(\tau)$ are possible and have been considered [34].

For long-term stability (τ of the order of months or even a year) one is severely limited in the size of m . In any case, m should be stated with any results.

F. Translations Among the Time-Domain Measures

For the five integral power laws shown on Table I, as well as intermediate values of α , two "bias functions" have been introduced by Barnes [35] to translate from $\langle \sigma_y^2(N_1, T_1, \tau_1) \rangle$ to $\langle \sigma_y^2(N_2, T_2, \tau_2) \rangle$ (the second definition given by (4.13) is used):

$$B_1 = \frac{\langle \sigma_y^2(N, T, \tau) \rangle}{\langle \sigma_y^2(2, T, \tau) \rangle} \quad (4.24)$$

and

$$B_2 = \frac{\langle \sigma_y^2(2, T, \tau) \rangle}{\langle \sigma_y^2(2, \tau, \tau) \rangle} \quad (4.25)$$

They have been computed and tabulated in [35].

Note that B_2 allows one to calculate the recommended measure $\sigma_y^2(\tau)$ from a two-sample variance measured with dead-time, provided the spectral density slope is known:

$$\sigma_y^2(\tau) = \frac{\langle \sigma_y^2(2, T, \tau) \rangle}{B_2} \quad (4.26)$$

This is of practical interest since counting techniques usually have nonzero dead-time between successive measurements.

V. TRANSLATIONS AMONG FREQUENCY-DOMAIN AND TIME-DOMAIN MEASURES: THE TRANSFER FUNCTION CONCEPT

To cover adequately the needs of oscillator users, two domains have been considered, namely the Fourier frequency and the (averaging) time domains. Moreover, several parameters have been introduced in each case. Translations among frequency domain measures and then among time domain measures were dealt with, respectively, in Sections III-C and IV-F.

It is now of prime importance to study the translations between the two domains for several reasons:

- 1) these relations provide a unified picture of frequency stability characterization;
- 2) the power laws which have been identified in the frequency domain may be translated into specific laws of $I(\tau)$ or $\sigma_y(\tau)$ through these relations;
- 3) stability measurements are sometimes made in only one domain due to equipment availability or capability: relations between parameters can then give an estimate of the performance in the other domain.

As will be demonstrated, a so-called "transfer function" appears in the general relationships between frequency domain and time-domain parameters.

This concept will be used in the following, either to study the properties of classical time-domain measures, or even to introduce new time-domain measures offering new measurement capabilities (Section VI).

A. True Variance Versus Spectral Density

First, the true variance $I^2(\tau)$ will be expressed in terms of $S_y(f)$ as

$$I^2(\tau) = \int_0^\infty S_y(f) |H_I(f)|^2 df \quad (5.1)$$

where $H_I(f)$ is a transfer function.

1) *Demonstration:* Equation (4.5) may be rewritten as

$$I^2(\tau) = \left\langle \left(\frac{1}{\tau} \int_{t_k}^{t_k + \tau} y(\theta) d\theta \right)^2 \right\rangle \quad (5.2)$$

or again

$$I^2(\tau) = \left\langle \left(\int_{-\infty}^\infty y(t) h_I(t_k - t) dt \right)^2 \right\rangle \quad (5.3)$$

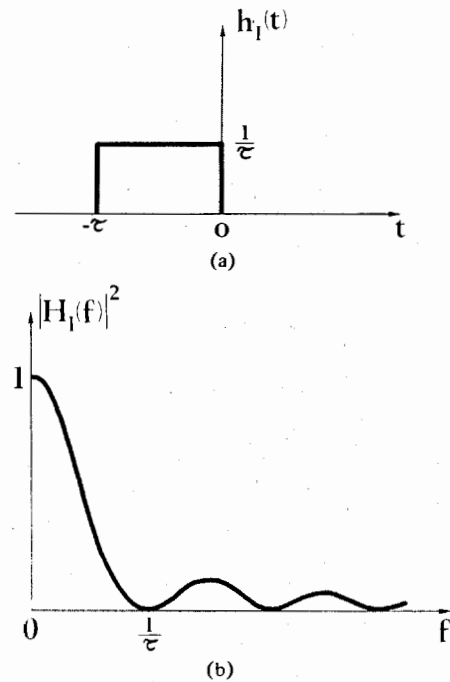


Fig. 6. (a) Measurement sequence of the true variance. (b) Transfer function of the true variance.

with

$$h_I(t) = \begin{cases} 0, & t < -\tau \\ \frac{1}{\tau}, & -\tau \leq t \leq 0 \\ 0, & t > 0. \end{cases} \quad (5.4)$$

Equation (5.3) includes a convolution integral where $y(t)$ is convolved with a function $h_I(t)$ which resembles the basic measurement sequence of one sample \bar{y}_k (Fig. 6(a)). This integral may represent the output signal of a hypothetical filter with impulse response $h_I(t)$ receiving an input signal $y(t)$: it is well known that the spectral density of the output signal is then given by the product $S_y(f) |H_I(f)|^2$ where the filter transfer function $H_I(f)$ is the Fourier transform of $h_I(t)$ and hence (Fig. 6(b)):

$$|H_I(f)|^2 = \left(\frac{\sin \pi \tau f}{\pi \tau f} \right)^2 \quad (5.5)$$

The true variance which is the mean-square value of this pseudo-output signal (5.3) is then equal to the area under its spectral density, i.e.,

$$I^2(\tau) = \int_0^\infty S_y(f) \left(\frac{\sin \pi \tau f}{\pi \tau f} \right)^2 df \quad (5.6)$$

Thus the true variance may be calculated from $S_y(f)$ through an integral including the transfer function $(\sin \pi \tau f / \pi \tau f)^2$ [3], [5], [6] and this is not restricted to power law spectral densities.

The above relationship cannot be reversed in a closed form in the most general cases, i.e., $S_y(f)$ cannot be expressed in terms of $I^2(\tau)$; this is an example of the key role played by spectral densities for random process characterization.

2) *Applications:* Some limitations of the usefulness of $I^2(\tau)$ appear from the fact that the transfer function is equal to one

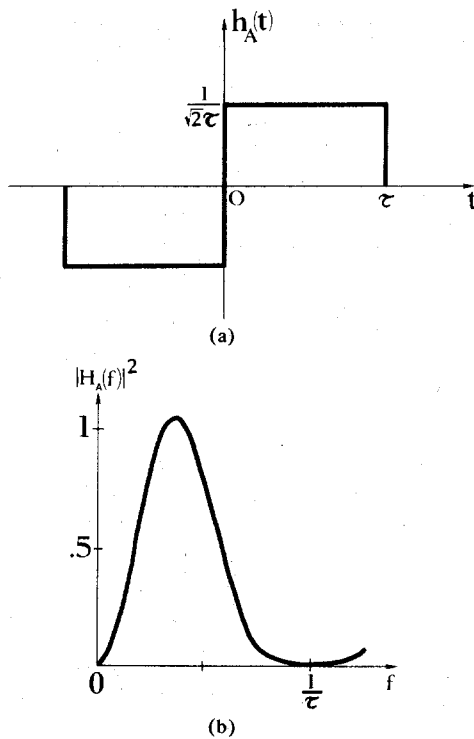


Fig. 7. (a) Measurement sequence of the two-sample variance. (b) Transfer function of the two-sample variance.

for $\pi\tau f \ll 1$: the true variance is therefore very sensitive to the low Fourier frequency components of $S_y(f)$, i.e., to the very slow fluctuations of the oscillator output frequency $\nu(t)$.

More specifically, a major problem arises for flicker frequency noise since the model $S_y(f) = h_{-1}f^{-1}$ yields $I^2(\tau) = \infty$. Thus the true variance is not suited to study real oscillators since most of them are perturbed by this noise [23]. Of course, this is a model pathology but the practical utility of $I^2(\tau)$ is also severely limited since actual measurements of $I^2(\tau)$ depend on the total number of samples and increase indefinitely with it when flicker frequency noise exists [5], [7]:

For this reason, the true variance has never been widely used as a practical stability measure, although it may be useful for theoretical work with well-behaved spectral density models. As an example, the RF spectrum $S_v(f)$ may be deduced from $I^2(\tau)$ when $\varphi(t)$ and $y(t)$ are Gaussian processes with $y(t)$ stationary [28]:

$$S_v^{(TS)}(f) = \frac{V_0^2}{2} \int_{-\infty}^{\infty} \exp\left(-\frac{\omega_0^2}{2} \tau^2 I^2(\tau)\right) \cos \omega_0 \tau \exp(-i2\pi f \tau) d\tau. \quad (5.7)$$

The laws found for white frequency noise and white phase noise are, respectively,

$$I^2(\tau) = \frac{h_0}{2\tau} \quad (5.8)$$

and

$$I^2(\tau) = \frac{h_2 f h}{2\pi^2 \tau^2} \left[1 - \frac{\sin 2\pi f h \tau}{2\pi f h \tau}\right]. \quad (5.9)$$

In the last case, f_h is necessary for convergence and $I(\tau) \sim \tau^{-1}$ for $2\pi f_h \tau \gg 1$.

For flicker phase noise, a HF cutoff is also needed for convergence and $I(\tau) \sim \tau^{-1}$ for $2\pi f_h \tau \gg 1$ [6]. Thus, both white and flicker phase noises provide the same slope on an $I(\tau)$ curve, and thus cannot be unambiguously recognized from such a plot.

B. Sample Variances Versus Spectral Density

Let us consider the second definition of the sample variance given by (4.13) since it has been recommended as a basis for time domain measures. The relation between its average value and $S_y(f)$ may be demonstrated very simply for adjacent samples⁸ [37]; combining the definitions (4.8) and (4.13) with the law expressed by equation (4.10):

$$\langle \sigma_{(2)}^2(N, \tau, \tau) \rangle = \frac{N}{N-1} [I^2(\tau) - I^2(N\tau)]. \quad (5.10)$$

The substitution of (5.6) into this relation gives:

$$\langle \sigma_y^2(N, \tau, \tau) \rangle = \frac{N}{N-1} \int_0^{\infty} S_y(f) \left(\frac{\sin \pi \tau f}{\pi \tau f}\right)^2 \left[1 - \left(\frac{\sin N \pi \tau f}{N \sin \pi \tau f}\right)^2\right] df. \quad (5.11)$$

The key point is the transfer function which behaves now as $[N(N+1)/3](\pi\tau f)^2$ for $N\pi\tau f \ll 1$.

Convergence of the integral at the lower limit is thus ensured even for flicker frequency noise and random walk frequency noise: the sample variance is thus a useful tool for time domain characterization. Greater values of N increase the sensitivity of the sample variance to low Fourier frequency components: in fact, the true variance is approached as $N \rightarrow \infty$.

C. Two-Sampled Variance Versus Spectral Density

Since $\sigma_y^2(\tau)$ is by far the most widely used time-domain stability measure, let us consider its relationship with $S_y(f)$ and apply it to the typical kinds of frequency variations encountered in real oscillators.

Making $N = 2$ in (5.11) gives directly

$$\sigma_y^2(\tau) = \int_0^{\infty} S_y(f) \frac{2 \sin^4 \pi \tau f}{(\pi \tau f)^2} df. \quad (5.12)$$

The transfer function is such that the integral converges at the lower limit for the five power laws reported in Table I. Here again, the relation cannot be inverted for the most general cases.

As shown on Fig. 7, the transfer function $|H_A(f)|^2$ appearing in (5.12) is the $|\text{Fourier Transform}|^2$ of a step-wise time function $h_A(t)$ that resembles the measurement sequence of the quantity $(\bar{y}_2 - \bar{y}_1)/\sqrt{2}$ involved in the definition of $\sigma_y^2(\tau)$ (4.18).

This general property will be used later to develop new time-domain parameters.

⁸A complete derivation was given by Cutler for nonadjacent samples [8]:

$$\langle \sigma_y^2(N, T, \tau) \rangle = \frac{N}{N-1} \int_0^{\infty} S_y(f) \left(\frac{\sin \pi \tau f}{\pi \tau f}\right)^2 \left[1 - \left(\frac{\sin N \pi T f}{N \sin \pi T f}\right)^2\right] df$$

1) *Application to the Power-Law Model:* The substitution of the widely used power-law spectral density model into (5.12) yields the expressions shown in Table II.

From these results, it appears that:

1) An upper cutoff frequency f_h is necessary for white and flicker phase noises since a substantial fraction of the noise "power" lies in the higher Fourier frequencies when $S_y(f)$ varies as f^2 or f . It has to be specified and the reported expressions are valid only for $2\pi f_h \tau \gg 1$.

2) $\sigma_y^2(\tau)$ obeys to power laws with a slight modification for flicker phase noise because of the logarithm. Thus a log-log plot of $\sigma_y(\tau)$ contains segments of straight lines whose slopes may be easily identified.

3) The results may be used to translate from time domain into the frequency domain for the power law model (3.9), although the general relationship (5.12) is not reversible. Since both white and flicker phase noises give very similar slopes, there is some ambiguity in noise identification when time-domain measurements give $\sigma_y(\tau) \sim \tau^{-1}$ (varying f_h purposely may help to distinguish between white and flicker phase noises [6]).

4) For flicker frequency noise, $\sigma_y(\tau)$ is independent of τ and the corresponding flat part on the plot is often referred to as the flicker floor [23].

For a given oscillator, $\sigma_y^2(\tau)$ is the sum of two or three terms, e.g., cesium beam standards are often well-modeled by

$$\sigma_y^2(\tau) = \frac{h_0}{2\tau} + 2 \ln 2h_{-1} \quad (5.13)$$

and the values of h_0 and h_{-1} may be deduced from measurements of $\sigma_y(\tau)$ over a sufficient range of τ .

5) $\sigma_y^2(\tau)$ can be a useful measure even when the model of (3.9) is not valid.

2) *Application to Sinusoidal FM:* Even the best sources are frequency modulated by unwanted sinusoidal signals. Although the above stability measures were developed to deal with random processes, sinusoidal instabilities do have an influence on them.

a) *Fourier frequency domain:* Assuming that

$$y(t) = \frac{\Delta\nu_0}{\nu_0} \sin 2\pi f_m t \quad (5.14)$$

the spectral density $S_y(f)$ contains a discrete line at the modulating frequency f_m :

$$S_y(f) = \frac{1}{2} \left(\frac{\Delta\nu_0}{\nu_0} \right)^2 \delta(f - f_m). \quad (5.15)$$

Looking for discrete lines in spectral densities provides a convenient means to identify periodic variations. Their presence usually does not interfere with the identification of the slopes due to random noise.

b) *Time domain:* The substitution of (5.15) into (5.12) gives⁹

$$\sigma_y^2(\tau) = \left(\frac{\Delta\nu_0}{\nu_0} \right)^2 \frac{\sin^4 \pi f_m \tau}{(\pi f_m \tau)^2} \quad (5.16)$$

Thus the effect of sinusoidal FM is zero when τ equals the

TABLE II

TWO-SAMPLE VARIANCE $\sigma_y^2(\tau)$ FOR THE POWER-LAW SPECTRAL DENSITY MODEL (ADAPTED FROM [8]; SEE ALSO [36]).

$S_y(f)$	$\sigma_y^2(\tau)$	Slope of $\sigma_y^2(\tau)$ VS. τ on log-log plot
$h_2 f^2$	$\frac{3 h_2 f_h}{4 \pi^2 \tau^2}$	-2
$h_1 f$	$\frac{h_1}{4 \pi^2 \tau^2} \left\{ 1.038 + 3 \ln 2 \pi f_h \tau \right\}$	≈ -2
h_0	$\frac{h_0}{2\tau}$	-1
$h_{-1} f^{-1}$	$2 \ln 2 h_{-1}$	0
$h_{-2} f^{-2}$	$\frac{2 \pi^2 h_{-2} \tau}{3}$	+1

modulation period $T_m = f_m^{-1}$ or one of its harmonics since the modulating signal is completely averaged out.

The worst case occurs when τ is near $T_m/2$ or one of its harmonics [19], [38].

As a practical consequence, when caution has not been exercised about the relation between the experimental values of τ and the expected value of T_m , there may be some scatter of the data because of the oscillating behavior of (5.16) (of course, this scatter is added to the law(s) due to the random noise(s) present).

3) *Application to Linear Frequency Drift:* Equation (5.12) is not useful for evaluating $\sigma_y^2(\tau)$ when linear frequency drift exists (i.e., $y(t) = d_1(t)$) since no tractable model seems to exist for $S_y(f)$ in this case.

Direct calculation in the time domain from (4.18) gives:

$$\sigma_y(\tau) = \frac{d_1}{\sqrt{2}} \tau. \quad (5.17)$$

Thus, linear frequency drift yields a τ^{+1} law for the square root of the two-sample variance. Such a law is observed for great values of τ when linear frequency drift has not been removed before statistical treatment. (Notice that no power law spectral density yields the same law; from Table II, it should be $S_y(f) = h_{-3} f^{-3}$ but $\sigma_y^2(\tau)$ diverges in this case due to its transfer function).

Higher order polynomial drifts will be treated in Section VIII in relation with structure functions.

VI. TRANSFER FUNCTIONS AS A TOOL TO DEFINE NEW PARAMETERS

Up to now, we have dealt only with the basics of frequency stability characterization.

1) Two domains have been considered and parameters have been defined independently in both domains.

2) Relationships have been established among these parameters that include, for each time-domain parameter, a specific transfer function which is the Fourier transform of the measurement sequence. Let us emphasize that the shapes of the transfer functions were not chosen *a priori*, but result from the definition of the time-domain parameters.

⁹Direct calculation through (4.18) and (5.14) gives the same result.

3) From the experimental point of view, spectrum analyzers are commonly associated with frequency-domain measurements and digital electronic counters with time-domain measurements.

In this section, the transfer function concept will be used as a tool to define new parameters giving new possibilities to existing measurement test sets.

A. The Hadamard Variance

The Hadamard variance has been developed by Baugh [39] for making high-resolution spectral analysis of $y(t)$ from measurements of \bar{y}_k , i.e., the frequency-domain parameter $S_y(f)$ is estimated from time-domain data provided by digital counters.

First, the limitations of the two-sample variance $\sigma_y^2(\tau)$ for such a spectral analysis must be emphasized and understood.

1) *Limitations of $\sigma_y^2(\tau)$ for Spectral Analysis:* Equation (5.12) shows that $\sigma_y^2(\tau)$ is not well suited for high-resolution spectral analysis of $y(t)$ since the main lobe of the transfer function (Fig. 7(b)) is wide. As a consequence, for a given value τ_c of the averaging time, the measured estimate $\hat{\sigma}_y^2(\tau_c)$ will not provide a precise estimate of $S_y(f_c)$ where $f_c \approx 0.37 \tau_c^{-1}$ is the main lobe center frequency for $\tau = \tau_c$.

For power-law spectral densities, the fact that both white phase noise and flicker phase noise yield nearly the same slope $\sigma_y(\tau) \sim \tau^{-1}$ results from this lack of selectivity.

However, let us remember that $\sigma_y(\tau)$ was introduced and recommended as a practical meaningful time-domain measure and not as a powerful spectral analysis tool.

2) *Definition of the Hadamard Variance:* Using the link between measurement sequences and transfer functions, the Hadamard variance is defined by a measurement sequence such that the corresponding transfer function contains a narrow lobe well-suited for spectral analysis [39]:

$$\langle \sigma_H^2(N, T, \tau) \rangle = \langle (\bar{y}_1 - \bar{y}_2 + \bar{y}_3 - \dots - \bar{y}_N)^2 \rangle. \quad (6.1)$$

The Hadamard variance¹⁰ is thus calculated from groups of N samples $\bar{y}_k (k = 1 \text{ to } N)$ and is related to $S_y(f)$ by:

$$\langle \sigma_H^2(N, T, \tau) \rangle = \int_0^\infty S_y(f) |H_H(f)|^2 df \quad (6.2)$$

where the transfer function $H_H(f)$ is the Fourier transform of the measurement sequence $h_H(t)$ (shown on Fig. 8(a) for $N = 6$ and $T = \tau$).

Sauvage [40] has shown that:

$$|H_H(f)|^2 = \left(\frac{\sin \pi T f}{\pi T f} \right)^2 \left(\frac{\sin N \pi T f}{\cos \pi T f} \right)^2. \quad (6.3)$$

The transfer function exhibits a narrow main lobe centered at the Fourier frequency $f_1 = (2T)^{-1}$; its bandwidth decreases with increasing N as demonstrated from the calculation of an equivalent bandwidth [40].

Before emphasizing the practical utility of $\langle \sigma_H^2(N, T, \tau) \rangle$, its limitations must be recognized and accounted for.

3) *Limitations of the Hadamard Variance:* Here again, the limitations are a consequence of the transfer function shape. Although it has a narrow main lobe, it contains also other unwanted features:

¹⁰ T and τ have the same meaning as for the sample variance. $\sigma_y^2(\tau)$ is one half times $\langle \sigma_H^2(N, \tau, \tau) \rangle$ with $N = 2$.

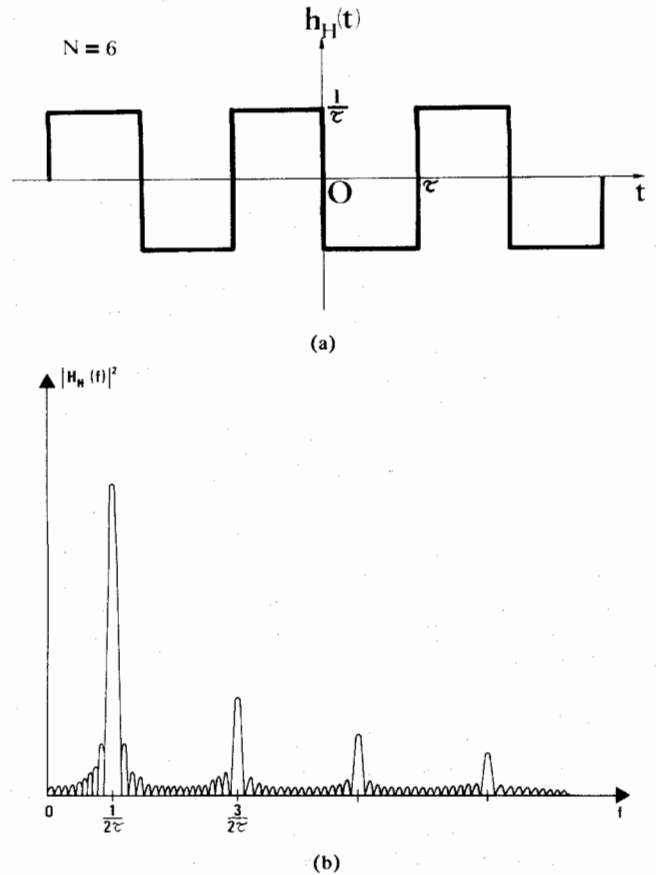


Fig. 8. (a) Measurement sequence of the Hadamard variance. (b) Transfer function of the Hadamard variance.

a) For adjacent samples ($T = \tau$), spurious responses appear at odd harmonics of the main lobe center frequency f_1 . When measurements of $\langle \sigma_H^2(N, \tau, \tau) \rangle$ are made, they all contribute to the measured value and thus may lead to important errors in the estimate of $S_y(f_1)$. For white phase noise where $S_y(f) = h_2 f^2$, each harmonic lobe yields the same contribution to the measured Hadamard variance as the main lobe (which ideally should be alone);

b) As shown on Fig. 8(b), the transfer function has also large sidelobes around the main lobe centered at f_1 .

4) *Improvements of the Hadamard Variance:* Limitation (a) can be partly eliminated with a dead time between successive samples of \bar{y}_k , i.e., with $T > \tau$. The optimum dead time $\tau_D = \tau/2$ eliminates the 3rd, 9th, 15th, etc., harmonic responses; in this case $T = 3\tau/2$.

More complicated measurement sequences have also been proposed to reduce the harmonic content wherein a pseudo-sinusoidal weighting of the samples \bar{y}_k suppresses the 3rd, 5th, 7th, and 9th harmonics [31], [40]. Analog filtering techniques have also been used to filter out the spurious harmonic responses [41].

Limitation b) may be partly or even completely eliminated by multiplying each \bar{y}_k within the set of N samples by an appropriate weighting factor. The unwanted sidelobes are completely suppressed by weighting the samples with binomial coefficients (BC) [39].

$$\langle \sigma_{HBC}^2(N, T, \tau) \rangle = \left\langle \left(\sum_{k=1}^N (-1)^{k-1} \binom{N-1}{k-1} \bar{y}_k \right)^2 \right\rangle \quad (6.4)$$

and the corresponding transfer function reads then as [31] [40]:

$$|H_{HBC}(f)|^2 = 2^{2(N-1)} \left(\frac{\sin \pi \tau f}{\pi \tau f} \right)^2 \sin^{2(N-1)} \pi T f. \quad (6.5)$$

5) *Practical Utility of the Hadamard Variance*: Provided adequate care has been exercised to overcome its limitations, the Hadamard variance is a useful tool for experiments since it extends frequency domain measurements down to very low Fourier frequencies of the order of 10^{-3} Hz just by increasing T and τ in the time domain, whereas analog spectrum analyzers are usually limited to about 1 to 10 Hz by hardware problems.

For the optimum dead time $\tau_D = \tau/2$, a working approximation of equation (6.2) giving an estimate of $S_y(f)$ reads as:

$$\hat{S}_y(f_1) \approx \frac{\tau}{N} \left\langle \sigma_H^2 \left(N, \frac{3\tau}{2}, \tau \right) \right\rangle \quad (6.6)$$

where $T = 3\tau/2$ and $f_1 = 1/2T = 1/3\tau$. Through this equation, spectral density estimates are made from measurements done with digital counters in the time domain. Of course, the Hadamard variance is itself estimated from a finite number of groups of N samples \bar{y}_k .

B. The Transfer Function Approach

In all the preceding approaches to time domain, any parameter, say $\sigma^2(\tau)$, was defined by its measurement sequence $h(t)$ (a step-wise function involving quantities such as N , T , τ , and possibly weighting factors) and related to $S_y(f)$ by

$$\sigma^2(\tau) = \int_0^\infty S_y(f) |H(f)|^2 df \quad (6.7)$$

where $H(f)$ is the Fourier transform of $h(t)$. The parameter was measured with a counter programmed to follow the chosen measurement sequence.

In the so-called "transfer function approach" developed by Rutman [42], the opposite point of view has been adopted: the parameter $\sigma^2(\tau)$ appearing in (6.7) is defined by the transfer function $H(f)$ whose shape is chosen *a priori*, even if no corresponding measurement sequence exists in the time domain (i.e., even if the inverse Fourier transform of $H(f)$ is not a step-wise function). One can thus consider a broader class of transfer functions since they need not be the Fourier transform of step-wise functions: however, counting techniques can no longer be used to measure the parameter defined in this way since the \bar{y}_k 's do not explicitly appear in this approach. The relevant measurement technique will be described.

To illustrate the usefulness of this approach, let us study two variances defined by their transfer functions, the "high-pass variance" and the "bandpass variance."

1) *The High-Pass Variance*: First, it is convenient to rewrite (6.7) as

$$\sigma^2(\tau) = \frac{8}{\omega_0^2 \tau^2} \int_0^\infty S_\varphi(f) |H_\varphi(f)|^2 df \quad (6.8)$$

since (5.12) can itself be rewritten as

$$\sigma_y^2(\tau) = \frac{8}{\omega_0^2 \tau^2} \int_0^\infty S_\varphi(f) \sin^4 \pi \tau f df. \quad (6.9)$$

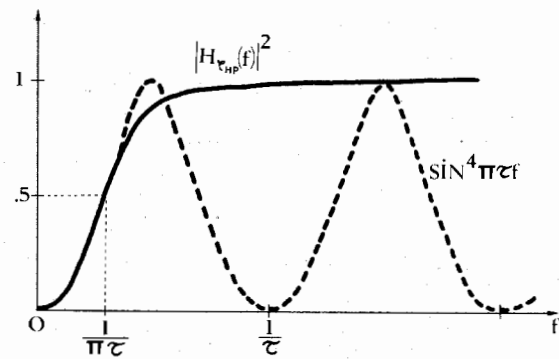


Fig. 9. Transfer function of the high-pass variance.

TABLE III
HIGH-PASS VARIANCE FOR THE POWER-LAW SPECTRAL DENSITY MODEL [42]

$S_y(f)$	$\sigma_{HP}^2(\tau)$	Slope vs. τ	$\frac{\sigma_{HP}(\tau)}{\sigma_y(\tau)}$
$h_2 f^2$	$\frac{2 h_2 f_h}{\pi^2 \tau^2}$	-2	1.63
$h_1 f$	$\frac{h_1}{2\pi^2 \tau^2} \left[\ln \left(1 + (\pi \tau f_h)^4 \right) \right]$	≈ -2	1.36 to 1.63 ($\pi \tau f_h = 10$ to ∞)
h_0	$\frac{h_0}{\sqrt{2} \tau}$	-1	1.19
$h_{-1} f^{-1}$	$\frac{\pi h_{-1}}{2}$	0	1.06
$h_{-2} f^{-2}$	$\frac{\pi^2 h_{-2} \tau}{\sqrt{2}}$	+1	1.03

* Convergence for higher negative slopes can be obtained by taking a higher-order high-pass Butterworth filter. Second-order was chosen here for greater similarity with $\sin^4 \pi \tau f$ for $\pi \tau f \ll 1$.

The subscript φ in $|H_\varphi(f)|^2$ indicates that $|H_\varphi(f)|^2$ multiplies the spectral density of phase noise. The new variances will be named according to the shape of their $|H_\varphi(f)|^2$.

The high-pass variance $\sigma_{HP}^2(\tau)$ is defined by (6.8) with a second-order high-pass Butterworth filter:

$$|H_{\varphi_{HP}}(f)|^2 = \frac{f^4}{f_c^4 + f^4} \quad (6.10)$$

where the cutoff frequency f_c is such that $|H_{\varphi_{HP}}(f_c)|^2 = 0.5$. The relation between f_c and the value of τ in $\sigma_{HP}^2(\tau)$ is chosen as $f_c = (\pi\tau)^{-1}$ by definition¹¹; the above $|H_{\varphi_{HP}}(f)|^2$ and $\sin^4 \pi \tau f$ appearing in equation (6.9) are illustrated on Fig. 9: they are very close for $\pi \tau f \lesssim 1$, but $|H_{\varphi_{HP}}(f)|^2$ is not periodic.

The expressions of $\sigma_{HP}^2(\tau)$ for the power-law spectral density model are reported in Table III and show that $\sigma_y^2(\tau)$ and $\sigma_{HP}^2(\tau)$ have the same general behavior: same laws versus τ , same dependence or independence versus f_h , same range of convergence for power laws (due to the choice of a second-order filter for $\sigma_{HP}^2(\tau)$), and same order of magnitude as shown by the ratio $\sigma_{HP}(\tau)/\sigma_y(\tau)$ which may be interpreted as a bias coefficient when $\sigma_{HP}(\tau)$ is used as an estimate of $\sigma_y(\tau)$.

¹¹ Notice that $\sin^4 \pi \tau f_c = 0.5$ for $f_c = (\pi\tau)^{-1}$.

From this similarity, one gets a better understanding on why $\sigma_y^2(\tau)$ is not well suited for high-resolution spectral analysis: despite the bandpass shape of each lobe, the $\sin^4 \pi \tau f$ transfer function has indeed a global high-pass behavior which is poorly selective (in particular, white phase noise and flicker phase noise cannot be completely resolved by this filter).

Moreover, the high-pass variance demonstrates clearly that instability (due to random noise) over a time interval τ is equally due to all Fourier frequency components of $S_\varphi(f)$ lying above $f_c = (\pi\tau)^{-1}$, e.g., above 318 Hz for $\tau = 1$ ms, (and under f_h). This result has been used to estimate $\sigma_y(\tau)$ from records of the RF spectrum assuming the low modulation index is valid (3.11) [28]; a working formula for white phase noise reads as

$$\sigma_y^2(\tau) \simeq \frac{3f_h 10^{-|\beta_0|/10}}{2\pi^2 \nu_0^2 \tau^2} \quad (6.11)$$

where β_0 denotes the RF spectrum level under the carrier expressed in dB/Hz. Formulas are given in [28] for other power-law noises.

2) *The Bandpass Variance*: Having recognized the lack of selectivity of $\sigma_y(\tau)$ and $\sigma_{HP}(\tau)$, one may use the transfer function approach to introduce a more selective variance through (6.8): the bandpass variance $\sigma_{BP}^2(\tau)$ defined by its second-order bandpass Butterworth transfer function with a center frequency $f_0 = (2\tau)^{-1}$ and a constant Q factor (by analogy with the first lobe of $\sin^4 \pi \tau f$ centered at $f = (2\tau)^{-1}$ and for which $Q \simeq 1.37$). Fig. 10 shows the new transfer function.

The expressions of $\sigma_{BP}^2(\tau)$ for the power-law model are reported in Table IV and show significant departures from those of $\sigma_y^2(\tau)$ or $\sigma_{HP}^2(\tau)$.

The higher selectivity of $|H_{\varphi BP}(f)|^2$ has the following consequences.

a) The high-frequency cutoff f_h is no longer necessary for convergence in the cases of white and flicker phase noises and therefore does not appear anymore in the relevant expressions of $\sigma_{BP}^2(\tau)$.

b) The slope of $\sigma_{BP}^2(\tau)$ takes a different value for each kind of noise including white phase and flicker phase noises for which it is equal to -3 and -2, respectively, (instead of -2 and ≈ -2 for $\sigma_y^2(\tau)$). For white phase noise, it is worth noting that $\sigma_{BP}(\tau)$ does not provide an estimate of $\sigma_y(\tau)$, even with a constant bias, since the slopes are different.

For the three last kinds of noise, $\sigma_{BP}(\tau)$ nevertheless provides an estimate of $\sigma_y(\tau)$ with bias factors near unity as shown in the last column.

To summarize, $\sigma_{BP}(\tau)$ should not be considered as an estimator of $\sigma_y(\tau)$ but rather as a useful tool in describing the statistical properties of oscillators since it reveals the presence of white and/or flicker phase noises on a time domain plot [43].

These properties of $\sigma_{BP}(\tau)$ are not surprising if we realize that the bandpass variance is indeed nothing but a constant percentage bandwidth (constant Q) spectral analysis of phase noise, the result of which is plotted in the time domain through (6.8).

Also, the bandpass variance may be viewed as an ideal Hadamard variance since the goal of any improvements on the latter is to get a transfer function with a single narrow lobe without any spurious responses: thus there is a conceptual link between the two approaches relying on the transfer function concept (of course, the measurement techniques are different as shown below).

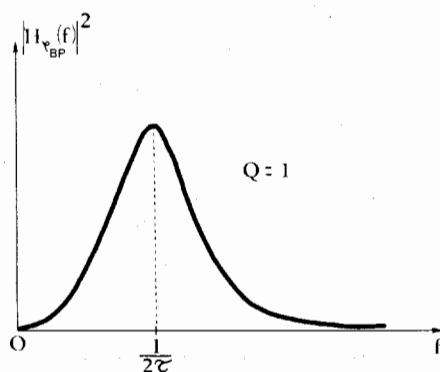


Fig. 10. Transfer function of the bandpass variance.

TABLE IV
BANDPASS VARIANCE FOR THE POWER-LAW SPECTRAL DENSITY MODEL [42]
($Q = 1$)

$S_y(f)$	$\sigma_{BP}^2(\tau)$	Slope VS. τ	$\frac{\sigma_{BP}(\tau)}{\sigma_y(\tau)}$
$h_2 f^2$	$\frac{h_2}{2\sqrt{2}\pi\tau^3}$	-3	$\frac{1.22}{\sqrt{\tau f_h}}$
$h_1 f$	$\frac{h_1}{\pi\sqrt{2}\tau^2}$	-2	$\frac{3}{\sqrt{1.038 + 3 \ln 2 \pi f_h \tau}}$
h_0	$\frac{\sqrt{2} h_0}{\pi\tau}$	-1	0.95
$h_{-1} f^{-1}$	$\frac{5 h_{-1}}{\pi\sqrt{2}}$	0	0.90
$h_{-2} f^{-2}$	$\frac{8\sqrt{2} h_{-2} \tau}{\pi}$	+1	0.74

3) *Practical Utility of the Transfer Function Approach*: Beyond its conceptual interest, the practical utility of this approach is in the additional possibilities it gives to the well-known phase detector measurement technique (see Section IX).

From (6.8), the parameter $\sigma(\tau)$ may be viewed as the rms value of the phase fluctuations $\varphi(t)$ filtered by a suitable transfer function $H_\varphi(f)$ are multiplied by the constant $\sqrt{8} (\omega_0 \tau)^{-1}$. Thus this approach leads to a data acquisition technique relying on a suitable filtering of the demodulated phase noise and therefore the phase detector technique may be used since it provides an analog output voltage proportional to the oscillator phase fluctuations [22], [44], [45].

An experimental test set designed for that purpose has been developed [43] and includes the following.

a) The well-known phase detector system with a loose phase-lock loop: the output voltage is proportional to the oscillator phase noise for Fourier frequencies greater than about 1 Hz when the loop time constant is about one second. Such a system is often used for frequency-domain measurements.

b) A filter whose transfer function $H_\varphi(f)$ may be adjusted to several shapes, especially high-pass and bandpass.

c) A true rms voltmeter as the final measurement apparatus. According to the filter response which has been selected, it is possible to measure the following.

- a) $S_\varphi(f)$ with a narrow bandpass filter, and this is indeed the classical application of this test set.
- b) $\sigma_{HP}(\tau)$ with a high-pass filter having a cutoff $f_c = (\pi\tau)^{-1}$; the upper cutoff frequency f_h is usually provided by another low-pass filter;
- c) $\sigma_{BP}(\tau)$ with a constant Q bandpass filter having a center frequency $f_0 = (2\tau)^{-1}$.

In the last two cases, $\sigma(\tau)$ is simply given by

$$\sigma(\tau) = \frac{\sqrt{8}}{k\omega_0\tau} v_{rms} \quad (6.12)$$

where k is a calibration constant and v_{rms} is the measured rms value of the filtered voltage analog to $\varphi(t)$.

Thus a unique experimental test set provides stability measures both in frequency and time domains, without any digital counter and associated statistical treatment of data in the latter case. Following the same basic idea, bandpass filtering of $y(t)$ at the output of a frequency discriminator has also been used to provide an estimate of $\sigma_y(\tau)$ [46].

VII. OTHER SUGGESTED TIME-DOMAIN MEASURES

In this section, we describe two more time-domain approaches relying on specific statistical treatments of the \bar{y}_k usually provided by counting techniques. In both cases, the new parameter is believed to be more "efficient" than $\sigma_y(\tau)$ in some respect to be discussed; the transfer function appears again as a convenient means to interpret their respective properties.

The links with the Hadamard variance and the sample variance will be given when appropriate.

The expressions of the new parameters will be presented for the power-law spectral density model allowing thus easier comparisons with the recommended time-domain measure $\sigma_y(\tau)$.

A. Finite-Time Frequency Control

This method introduced by Boileau and Picinbono [13] is based upon a combination of the \bar{y}_k that may be interpreted as a modified sample variance.¹² To illustrate this approach, a specific modified sample variance $\Sigma_y^2(\tau)$ has been considered in [47], the expressions of which are given below in Table V for the power-law spectral density model.

1) *The Modified Sample Variance:* From N discrete values of \bar{y}_k , the following quantity is defined as a modified sample variance:

$$\Sigma_y^2(N, T, \tau) = \left(\bar{y}_{(N+1)/2} - \frac{1}{N} \sum_{i=1}^N \bar{y}_i \right)^2 \quad (7.1)$$

where N is odd and $\bar{y}_{(N+1)/2}$ denotes the central sample within a set of N samples.

T and τ have the same meaning as in Fig. 4. Of course, $\Sigma_y^2(N, T, \tau)$ is a random variable and its infinite time average is related to $S_y(f)$ by the following integral [13]:

$$\Sigma_y^2(N, T, \tau) / \tau = \int_0^\infty S_y(f) \left(\frac{\sin \pi\tau f}{\pi\tau f} \right)^2 \left(1 - \frac{\sin N\pi T f}{N \sin \pi T f} \right)^2 df \quad (7.2)$$

¹²Continuous expressions were also developed in [13] and reviewed in [47]. Only the discrete realization based on discrete values of \bar{y}_k is presented here.

TABLE V
MODIFIED THREE-ADJACENT-SAMPLE VARIANCE $\Sigma_y^2(\tau)$ FOR THE POWER-LAW SPECTRAL DENSITY MODEL ($\pi\tau f_h \gg 1$)

$S_y(f)$	$\Sigma_y^2(\tau)$	Slope vs. τ	$\frac{\Sigma_y(\tau)}{\sigma_y(\tau)}$
$h_2 f^2$	$\frac{5 h_2 f_h}{9 \pi^2 \tau^2}$	-2	0.86
$h_1 f$	$\frac{5 h_1}{9 \pi^2 \tau^2} \left[.964 + \ln \pi\tau f_h \right]$	≈ -2	0.85 to 0.86 ($\pi\tau f_h = 10$ to ∞)
h_0	$\frac{h_0}{3\tau}$	-1	0.82
$h_{-1} f^{-1}$	$\frac{8 \ln 2 - 3 \ln 3}{3} h_{-1}$	0	0.74
$h_{-2} f^{-2}$	$\frac{2}{9} \pi^2 h_{-2} \tau$	+1	0.58
$h_{-3} f^{-3}$	$\frac{27 \ln 3 - 32 \ln 2}{9} \pi^2 h_{-3} \tau^2$	+2	-
$h_{-4} f^{-4}$	$\frac{44}{90} \pi^4 h_{-4} \tau^3$	+3	-

The new transfer function behaves now as $(N^2 - 1)^2 (\pi T f)^4 / 36$ for $N\pi T f \ll 1$, i.e., for the lowest Fourier frequencies. Due to this behavior, this time-domain parameter is convergent for the power-law model $S_y(f) = h_\alpha f^\alpha$ even for $\alpha = -3$ and $\alpha = -4$ (whereas the classical sample variance was limited to $\alpha = -2$). It is interesting to note that the law $S_y(f) = h_{-3} f^{-3}$ has been referred to as a possible model for a high-level quartz-crystal oscillator at Fourier frequencies around 1 Hz [21].

Thus the finite-time frequency-control method allows one to deal with spectral densities having greater negative slopes (sometimes interpreted as "stronger nonstationarity") than the classical sample variance approach.

2) *The Modified Three-Adjacent-Sample Variance:* For the same reasons as it appeared useful to recommend the use of one particular sample variance $\sigma_y^2(\tau)$, comparability of data will be improved by specifying the values of N and T in $\langle \Sigma_y^2(N, T, \tau) \rangle$, with τ remaining the independent variable [47].

Let us consider the modified sample variance with $T = \tau$ (adjacent samples as in $\sigma_y^2(\tau)$) and $N = 3$ (the smallest possible value of N for this measurement sequence where N is odd).¹³ Equations (7.1) and (7.2) then read as:

$$\Sigma_y^2(3, \tau, \tau) = \frac{1}{9} (2\bar{y}_2 - \bar{y}_1 - \bar{y}_3)^2 \quad (7.3)$$

$$\Sigma_y^2(\tau) \equiv \langle \Sigma_y^2(3, \tau, \tau) \rangle = \int_0^\infty S_y(f) \frac{16}{9} \frac{\sin^6 \pi\tau f}{(\pi\tau f)^2} df \quad (7.4)$$

The transfer function is the Fourier transform of the time-domain measurement sequence as shown on Fig. 11.

¹³A so-called curvature variance has been introduced by Kramer [60], [61], which is equal to 1.5 times $\Sigma_y^2(\tau)$. The second difference of \bar{y}_k or the third difference of φ_k was also discussed by Barnes in [30] and [69].

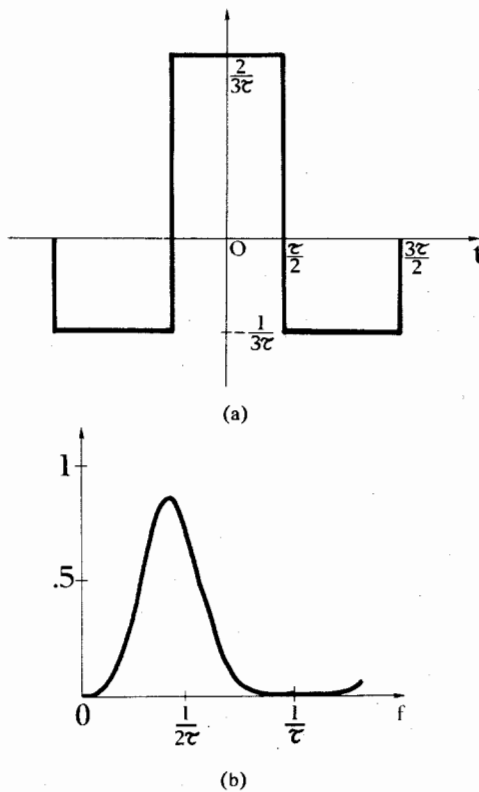


Fig. 11. (a) Measurement sequence of the modified three sample variance. (b) Transfer function of the modified three sample variance.

Table V gives the expressions of $\Sigma_y^2(\tau)$ for the power-law model, including the cases $\alpha = -3$ and $\alpha = -4$ for which $\sigma_y(\tau) = \infty$. For the five other encountered laws, $\Sigma_y(\tau)$ is an estimator of $\sigma_y(\tau)$ with constant bias coefficients reported in the right column: both follow the same laws vs τ and f_h and have the same order of magnitude. Especially, $\Sigma_y(\tau)$ is not better suited than $\sigma_y(\tau)$ to distinguish between white phase noise and flicker phase noise that yield nearly the same slope.

3) *Relations with Previous Approaches*: First, it must be recognized that $\Sigma_y^2(3, \tau, \tau)$ is quite different from $\sigma_y^2(3, \tau, \tau)$ derived from (4.13) with $N = 3$ and $T = \tau$:

$$\sigma_y^2(3, \tau, \tau) = \frac{1}{3} (\bar{y}_1^2 + \bar{y}_2^2 + \bar{y}_3^2 - \bar{y}_1 \bar{y}_2 - \bar{y}_1 \bar{y}_3 - \bar{y}_2 \bar{y}_3). \quad (7.5)$$

However, the Hadamard variance weighted by binomial coefficients (6.4) with $N = 3$ and $T = \tau$ reads as

$$\langle \sigma_{HBC}^2(3, \tau, \tau) \rangle = \langle (\bar{y}_1 - 2\bar{y}_2 + \bar{y}_3)^2 \rangle \quad (7.6)$$

and hence

$$\Sigma_y^2(\tau) = \frac{1}{9} \langle \sigma_{HBC}^2(3, \tau, \tau) \rangle. \quad (7.7)$$

It is easy to verify that making $N = 3$ and $T = \tau$ in (6.5) and then dividing by 9 yields the transfer function appearing in the above integral (7.4).

This link with the weighted Hadamard variance will be interpreted in the next section from the study of structure functions of phase.

As far as the transfer function approach is concerned, expression (7.4) may be rewritten in terms of the spectral density of

phase noise:

$$\Sigma_y^2(\tau) = \frac{64}{9\omega_0^2 \tau^2} \int_0^\infty S_\varphi(f) \sin^6 \pi \tau f df. \quad (7.8)$$

Thus $\Sigma_y^2(\tau)$ can be estimated by high-pass filtering the demodulated phase noise, but a third-order Butterworth filter is needed to yield the same convergence properties as the $\sin^6 \pi \tau f$ transfer function.

To summarize, the new parameter $\Sigma_y(\tau)$ whose experimental estimation by counting techniques is not very complicated, has the same behavior as $\sigma_y(\tau)$ for power-law spectral densities, except that it converges for two more negative slopes (one of which has been mentioned in [21]); therefore, its use may be recommended whenever a law such as $S_y(f) = h_{-3} f^{-3}$ (or $h_{-4} f^{-4}$) is expected from some theoretical and/or experimental consideration; then, variations of $\Sigma_y(\tau) \sim \tau$ (or $\tau^{3/2}$) may be observed for the greater values of τ . Note that these laws cannot be observed for $\sigma_y(\tau)$ since $+1/2$ is the highest positive slope that arises with power-law models; this point will be discussed in greater detail in Section VIII in connection with the effects of polynomial drifts.

B. Frequency Instability Over a Time Interval T

In all the preceding time domain approaches, the parameter of interest ($I(\tau)$, $\sigma_y(\tau)$, $\sigma_{HP}(\tau)$, $\sigma_{BP}(\tau)$, or $\Sigma_y(\tau)$) was plotted versus the averaging time interval τ which may vary from say 10^{-3} s. to thousands of seconds or even days or months. The decreasing parts of the curves show then the averaging of stationary random noises and, beyond the flicker floor, the increasing parts are due to nonstationary effects (either random or deterministic).

Another point of view has been proposed by De Prins and Cornelissen [48] who studied the frequency fluctuations over a time interval T for a fixed value of the averaging time interval τ : the parameter is thus plotted versus T (which has in this paragraph the same meaning as in Fig. 4).

Let us consider first the case $\tau = 0$ for simplicity and then the case $\tau \neq 0$ which is physically realistic.

1) *Zero Averaging Time ($\tau = 0$)*: An ideal sampling of $y(t)$ at t_k ($k = 1, 2, 3, \dots$) such that $t_{k+1} - t_k = T$ yields the instantaneous values $y(t_k)$ denoted as y_k . This idealized measurement process corresponds to instantaneous frequency measurements ($\tau = 0$).

The difference $\delta y_T = y_2 - y_1$ is a measure of the instantaneous fractional frequency instability over the time interval T from t_1 to t_2 . This is of course a random quantity whose mean-square value may be used as a statistical meaningful measure of frequency instability over T :

$$\langle \delta y_T^2 \rangle = \langle (y_2 - y_1)^2 \rangle. \quad (7.9)$$

Since only one sample of $y(t)$ is usually available to the experimenter, a suitable measurement process has been recommended [49] to avoid artefacts due to correlation between data.

This parameter is simply related to $R_y(\tau)$ and $S_y(f)$ by

$$\langle \delta y_T^2 \rangle = 2[R_y(0) - R_y(T)] \quad (7.10)$$

$$\langle \delta y_T^2 \rangle = \int_0^\infty S_y(f) 4 \sin^2 \pi T f df. \quad (7.11)$$

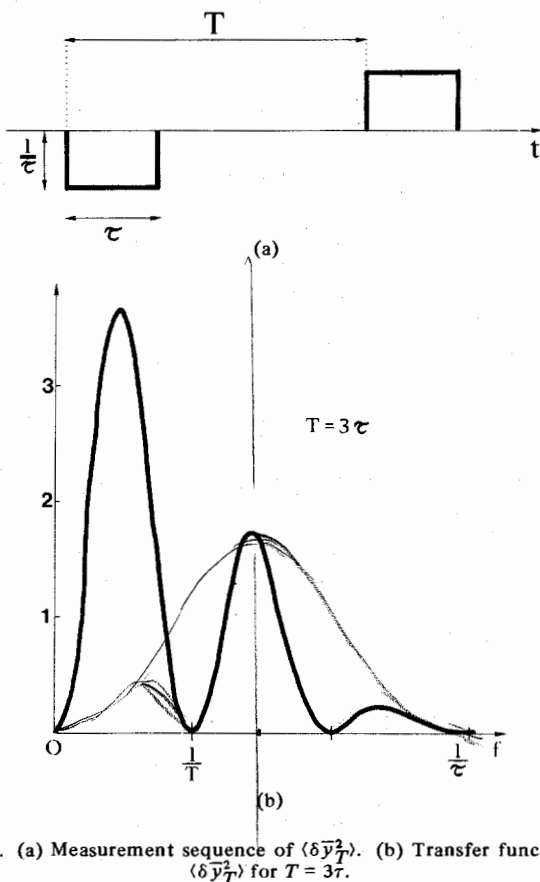


Fig. 12. (a) Measurement sequence of $\langle \delta \bar{y}_T^2 \rangle$. (b) Transfer function of $\langle \delta \bar{y}_T^2 \rangle$ for $T = 3\tau$.

Assuming that $y(t)$ is a zero-mean stationary process with finite variance $\langle y^2(t) \rangle = R_y(0)$, the following limit holds (see (7.10)):

$$\lim_{T \rightarrow \infty} \langle \delta y_T^2 \rangle = 2 \langle y^2(t) \rangle. \quad (7.12)$$

Assuming that $S_y(f)$ has an upper frequency cutoff f_h , it is easy to show from (7.11) that:

$$\langle \delta y_T^2 \rangle \simeq 4\pi^2 \beta_y T^2, \quad \text{for } \pi T f_h \ll 1 \quad (7.13)$$

where β_y is a constant equal to¹⁴

$$\beta_y = \int_0^{f_h} f^2 S_y(f) df. \quad (7.14)$$

Thus the evolution of $\langle \delta y_T^2 \rangle$ with T is quite different from the laws of any other time-domain measure versus the averaging time interval τ and this may be traced to the shape of the periodic transfer function appearing in (7.11). This evolution may be said to be "cumulative" in the sense that all the possible values of $y(t)$ are scanned and contribute to $\langle \delta y_T^2 \rangle$ as T increases (instead of being averaged out with increasing τ).

2) *Nonzero Averaging Time ($\tau \neq 0$):* Since experimental test sets provide averaged samples \bar{y}_k (4.2), it is of practical interest to generalize the definition of $\langle \delta y_T^2 \rangle$ as follows:

$$\langle \delta \bar{y}_T^2 \rangle = \langle (\bar{y}_2 - \bar{y}_1)^2 \rangle. \quad (7.15)$$

The measurement sequence and transfer function are shown on

¹⁴ For a power law $S_y(f) = h_\alpha f^\alpha$, the constant equals $\beta_{y,\alpha} = h_\alpha f_h^{\alpha+3} / (\alpha+3)$.

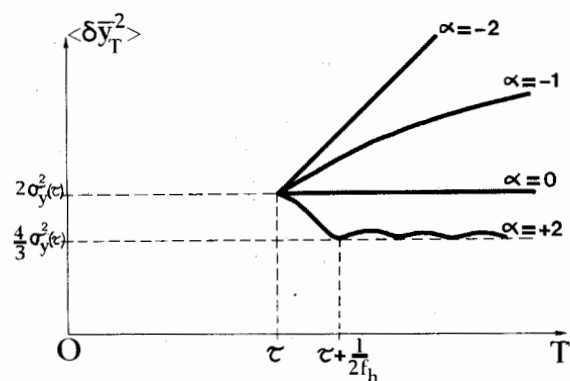


Fig. 13. The time-domain parameter $\langle \delta \bar{y}_T^2 \rangle$ versus T for the power law model.

Fig. 12, and the relationship with $S_y(f)$ now reads as

$$\langle \delta \bar{y}_T^2 \rangle = \int_0^\infty S_y(f) 4 \left(\frac{\sin \pi T f}{\pi T f} \right)^2 \sin^2 \pi T f df. \quad (7.16)$$

As shown below, this parameter is simply related to one particular sample variance.

3) *Relation Between $\langle \delta \bar{y}_T^2 \rangle$ and the Sample Variance:* From the following definition of the sample variance

$$\sigma_y^2(N, T, \tau) = \frac{1}{N-1} \sum_{i=1}^N \left(\bar{y}_i - \frac{1}{N} \sum_{j=1}^N \bar{y}_j \right)^2 \quad (7.17)$$

it is easy to demonstrate that

$$\langle \delta \bar{y}_T^2 \rangle = 2 \langle \sigma_y^2(2, T, \tau) \rangle. \quad (7.18)$$

In other words, $\langle \delta \bar{y}_T^2 \rangle$ is just twice the two-sample variance with nonadjacent samples ($T > \tau$) which will be here studied versus T for a fixed value of τ . Thus (7.16) is a particular case of the general formula given by Cutler (equation (23) with $N = 2$ in [8]; see also footnote 8).

Except for white phase noise for which a more complete formula is given here, the expressions of $\langle \delta \bar{y}_T^2 \rangle$ reported in Table VI are thus deduced from $\langle \sigma_y^2(2, T, \tau) \rangle$ given in [8], Appendix II.

For $T = \tau$, these expressions are equal to $2\sigma_y^2(\tau)$ as expected from (7.18), except the one for flicker phase noise valid only for $T \gg \tau$. For increasing values of T , the evolution of $\langle \delta \bar{y}_T^2 \rangle$ is pictured on Fig. 13: it is linear for random walk frequency noise, logarithmic for flicker frequency noise, independent of T for white frequency noise and it approaches the limit value of $(4/3) \cdot \sigma_y^2(\tau)$ for white and flicker phase noises (this evolution also appears on [35], Fig. 2 where the bias function B_2 (4.25) is plotted for $0 < T \leq 2\tau$).

These expressions have been used by Cornelissen [49] to predict the long-term behavior of frequency standards.

For real oscillators wherein flicker frequency noise is ever present, a logarithmic law of $\langle \delta \bar{y}_T^2 \rangle$ versus T may be expected for the greater values of T (when not masked by random walk of frequency or by deterministic drifts). In particular, linear frequency drift modeled by $y(t) = d_1 t$ yields:

$$\langle \delta \bar{y}_T^2 \rangle = d_1^2 T^2 \quad (7.19)$$

To summarize, despite its simple mathematical relationship with the two-nonadjacent sample variance, the parameter $\langle \delta \bar{y}_T^2 \rangle$ has been introduced on a quite different conceptual

TABLE VI
THE TIME-DOMAIN PARAMETER $\langle \delta y_T^2 \rangle$ FOR THE POWER-LAW SPECTRAL DENSITY MODEL

$S_y(f)$	$\langle \delta y_T^2 \rangle$ for $T > \tau$	
$h_2 f^2$	$\frac{h_2 f_h}{\pi^2 \tau^2} \left(1 + \frac{1}{2} \cdot \frac{\sin 2\pi (T-\tau) f_h}{2\pi (T-\tau) f_h} \right)$	for $\pi \tau f_h \gg 1$
$h_1 f$	$\frac{h_1}{\pi^2 \tau^2} \left(0.577 + \ln 2\pi f_h \tau + \frac{1}{2} \ln \frac{(T/\tau)^2}{(T/\tau)^2 - 1} \right)$	for $\pi \tau f_h \gg 1$ and $T \gg \tau$
h_0	$\frac{h_0}{\tau}$	
$h_{-1} f^{-1}$	$h_{-1} \left[-2 \left(\frac{T}{\tau} \right)^2 \ln \frac{T}{\tau} + \left(\frac{T}{\tau} + 1 \right)^2 \ln \left(\frac{T}{\tau} + 1 \right) + \left(\frac{T}{\tau} - 1 \right)^2 \ln \left(\frac{T}{\tau} - 1 \right) \right]$	$\approx 2 h_{-1} \ln \frac{T}{\tau}$ for $T \gg \tau$
$h_{-2} f^{-2}$	$2\pi^2 h_{-2} \left(T - \frac{\tau}{3} \right)$	

basis where emphasis is put on the time evolution of the fluctuations under study. Therefore, it may appear more useful than $\sigma_y(\tau)$ for certain kinds of prediction problems [49], and exhibits unfamiliar laws reported on Table VI.

VIII. THE STRUCTURE FUNCTION APPROACH

In the preceding sections, the discrete samples \bar{y}_k provided by a suitable measurement process have been used to define the following quantities that serve several purposes:

- 1) the true variance given by (4.5), as a theoretical measure of time-domain frequency instability;
- 2) the sample variances given by (4.8), (4.13), or (4.16), as practical measures of time-domain instability; $\sigma_y^2(\tau)$ given by (4.18) is just a particular case;
- 3) the Hadamard variance (6.1) or (6.4), as a time-domain measure leading to spectral density estimates;
- 4) the modified sample variance (7.1), as a time-domain measure which converges even for $S_y(f) \sim f^{-3}$ and f^{-4} , with $\Sigma_y^2(\tau)$ given by (7.4) being a particular case;
- 5) $\langle \delta \bar{y}_T^2 \rangle$ given by (7.11) as a different means to present instability data in the time domain.

In the original references, these parameters have been introduced to serve one of the above mentioned purposes but no unified presentation had been looked for.

The structure function approach developed by Lindsey and Chie [50], [51] plays such a unifying role in the sense that the most important time-domain parameters appear as particular cases of one general concept, namely the so-called structure function.

This theory is presented below with special emphasis on its relationships with previous approaches and on its application to deterministic polynomial drifts.

A. Definition of Structure Functions

For $M \geq 1$, the M th increment of a random process $g(t)$ is defined by

$$\Delta^{(M)}g(t; \tau) = \sum_{k=0}^M (-1)^k \binom{M}{k} g(t + (M-k)\tau). \quad (8.1)$$

The process $g(t)$ has a stationary M th increment [15], [16], if

the following averages exist for all real T and τ and do not depend on the instant t :

$$\langle \Delta^{(M)}g(t; \tau) \rangle = \alpha(\tau) \quad (8.2)$$

$$\langle \Delta^{(M)}g(t; \tau) \cdot \Delta^{(M)}g(t+T; \tau) \rangle = D_g^{(M)}(T; \tau). \quad (8.3)$$

In other words, the M th increment has a time independent mean and its autocorrelation depends only on the time difference (wide sense stationarity).

By definition, $D_g^{(M)}(T; \tau)$ is the structure function of the M th increment. In the following, the M th structure function of the random process $g(t)$ will be defined as $D_g^{(M)}(T=0; \tau)$ and denoted as $D_g^{(M)}(\tau)$:

$$D_g^{(M)}(\tau) = \langle (\Delta^{(M)}g(t; \tau))^2 \rangle. \quad (8.4)$$

B. Application to Oscillators Instabilities: Phase Instability Versus Frequency Instability

When dealing with instabilities in quasi-sinusoidal oscillators, the structure function concept may be applied to the phase fluctuations $\varphi(t)$ and to the (fractional) frequency fluctuations $y(t)$, yielding two sets of parameters [50]. Distinct definitions may thus be given for phase instability and frequency instability, whereas these two concepts have often been confused in the literature. To illustrate this point, let us consider the first ($M=1$) structure functions of $\varphi(t)$ and of $y(t)$.

1) *Phase Instability*: Phase instability over a time interval τ may be defined as the ratio of the rms value of the phase noise accumulated during τ seconds to the phase accumulated by a noiseless oscillator ($\omega_0 \tau$). This quantity is simply related to the first structure function of phase noise:

$$\frac{\sqrt{\langle [\varphi(t+\tau) - \varphi(t)]^2 \rangle}}{\omega_0 \tau} = \frac{1}{\omega_0 \tau} \sqrt{D_\varphi^{(1)}(\tau)}. \quad (8.5)$$

But (4.4) and (4.5) show that the above quantity is indeed nothing but the true standard deviation $I(\tau)$ usually interpreted as a theoretical measure of frequency instability (the limitations of which have been pointed out).

2) *Frequency Instability*: Fractional frequency instability over a time interval T may be defined as the rms value of the fractional frequency fluctuations $y(t)$ accumulated during T

TABLE VII
STRUCTURE FUNCTIONS OF PHASE: RELATIONS WITH OTHER TIME-DOMAIN PARAMETERS

M	$\Delta^{(M)} \varphi(t; \tau)$	$D_{\varphi}^{(M)}(\tau)$	Relations with other measures
1	$\omega_0 \tau \bar{y}_1$	$\omega_0^2 \tau^2 \langle \bar{y}_1^2 \rangle$	$I(\tau) = \frac{1}{\omega_0 \tau} \sqrt{D_{\varphi}^{(1)}(\tau)}$
2	$\omega_0 \tau (\bar{y}_2 - \bar{y}_1)$	$\omega_0^2 \tau^2 \langle (\bar{y}_2 - \bar{y}_1)^2 \rangle$	$\sigma_y(\tau) = \frac{1}{\sqrt{2} \omega_0 \tau} \sqrt{D_{\varphi}^{(2)}(\tau)}$
3	$\omega_0 \tau (\bar{y}_1 - 2\bar{y}_2 + \bar{y}_3)$	$\omega_0^2 \tau^2 \langle (\bar{y}_1 - 2\bar{y}_2 + \bar{y}_3)^2 \rangle$	$\Sigma_y(\tau) = \frac{1}{3 \omega_0 \tau} \sqrt{D_{\varphi}^{(3)}(\tau)}$
M	$\omega_0 \tau \sum_{i=1}^M (-1)^i \binom{M-1}{i-1} \bar{y}_i$	$\omega_0^2 \tau^2 \langle \left(\sum_{i=1}^M (-1)^i \binom{M-1}{i-1} \bar{y}_i \right)^2 \rangle$	$\langle \sigma_{HBC}^2(M, \tau, \tau) \rangle^{1/2} = \frac{1}{\omega_0 \tau} \sqrt{D_{\varphi}^{(M)}(\tau)}$

seconds; this is just the first structure function of $y(t)$ [52]:

$$\sqrt{\langle [y(t+T) - y(t)]^2 \rangle} = \sqrt{D_y^{(1)}(T)}. \quad (8.6)$$

Moreover, it is equal to $\sqrt{\langle \delta y_T^2 \rangle}$ defined in (7.9); that is why we kept the symbol T instead of τ in (8.6).

The parameter $\langle \delta y_T^2 \rangle$ defined by (7.11) is related to phase structure functions as follows [47]:

$$\langle \delta y_T^2 \rangle = \frac{2}{\omega_0^2 \tau^2} [D_{\varphi}^{(1)}(\tau) - D_{\varphi}^{(1)}(T; \tau)]. \quad (8.7)$$

When dealing with \bar{y}_k , structure functions of phase appear because of the link between \bar{y}_k and the first increment of $\varphi(t)$ expressed by (4.4). Structure functions of phase are hence of prime importance for oscillators instability characterization.

C. Structure Function of Phase

First, it must be remembered that taking the M th increments of the phase had been proposed by Barnes [30] as a powerful means of classifying the statistical fluctuations in oscillators: Structure functions of phase are just another way of rewriting these quantities.

1) *M*th Increment of Phase Versus \bar{y}_k : Since the \bar{y}_k are meaningful data, it is of interest to rewrite $\Delta^{(M)}\varphi(t; \tau)$ in terms of the M values of \bar{y}_k involved. By definition, $\Delta^{(M)}\varphi(t; \tau)$ is calculated from the instantaneous values of $\varphi(t)$ at the instants $t, t + \tau, t + 2\tau, \dots, t + M\tau$. The i th value $\varphi(t + (i-1)\tau)$, where $i = 1$ to $M+1$, is weighted by $(-1)^i \binom{M}{i-1}$ when M is odd and by $(-1)^{i+1} \binom{M}{i-1}$ when M is even; the term $\varphi(t + M\tau)$ has always a positive sign from the definition of $\Delta^{(M)}\varphi(t; \tau)$.

By using relation (4.4), $\Delta^{(M)}\varphi(t; \tau)$ may then be rewritten in terms of the M values of \bar{y}_k (with $k = 1$ to M) involved in its definition [47]:

$$\Delta^{(M)}\varphi(t; \tau) = \omega_0 \tau \sum_{k=1}^M (-1)^{k+1} \binom{M-1}{k-1} \bar{y}_k, \quad M \text{ odd} \quad (8.8)$$

$$\Delta^{(M)}\varphi(t; \tau) = \omega_0 \tau \sum_{k=1}^M (-1)^k \binom{M-1}{k-1} \bar{y}_k, \quad M \text{ even.} \quad (8.9)$$

These expressions are used next to relate $D_{\varphi}^{(M)}(\tau)$ to some of the time-domain measures.

2) *Interpretation of $D_{\varphi}^{(M)}(\tau)$ in Terms of Time-domain Measures*: From (8.8) and (8.9), the expressions of $D_{\varphi}^{(M)}(\tau)$ calculated for $M = 1, 2, 3$, etc., are reported in Table VII and show that some of the previous time domain measures may be re-interpreted in terms of structure functions of phase.

For $M = 1$ and $M = 2$, the structure function is related to the true variance $I^2(\tau)$ and to $\sigma_y^2(\tau)$, respectively, as pointed out by Lindsey and Chie [50].

For $M = 3$ and for M even (≥ 4), it is related respectively to $\Sigma_y^2(\tau)$ and to the Hadamard variance with adjacent samples weighted by binomial coefficients as pointed out in [47].

The unifying role of the structure function concept is then clear from the right column of Table VII.

3) *Relation Between $D_{\varphi}^{(M)}(\tau)$ and $S_y(f)$* : The relation between structure functions of phase and $S_y(f)$ may be easily deduced from (6.5) with $T = \tau$ and from the relation between $D_{\varphi}^{(M)}(\tau)$ and $\langle \sigma_{HBC}^2(M, \tau, \tau) \rangle$ reported in Table VII:

$$D_{\varphi}^{(M)}(\tau) = 2^{2(M-1)} (\omega_0 \tau)^2 \int_0^{\infty} S_y(f) \frac{\sin^{2M} \pi \tau f}{(\pi \tau f)^2} df. \quad (8.10)$$

The corresponding inverse transform relationship can be given in terms of Mellin transforms as shown in [50]; however, the formula involved are rather complex to use.

Because of the exponent $2M$ appearing in the transfer function, the desirability of taking structure functions with higher orders becomes obvious from the point of view of convergence with power-law spectral densities having steep negative slopes: this general relationship in connection with the expressions reported in Table VII explains why $\Sigma_y(\tau)$ is "more convergent" than $\sigma_y(\tau)$ which is itself "more convergent" than $I(\tau)$.

When $S_{\varphi}(f)$ may be defined unambiguously, equation (8.10) may be rewritten as

$$D_{\varphi}^{(M)}(\tau) = 2^{2M} \int_0^{\infty} S_{\varphi}(f) \sin^{2M} \pi \tau f df. \quad (8.11)$$

4) *Application of $D_{\varphi}^{(M)}(\tau)$ to Polynomial Drifts*: Up to now, we have been concerned mainly with the characterization of

TABLE VIII
TIME-DOMAIN PARAMETERS FOR POLYNOMIAL PHASE DRIFT

		ORDER OF THE PHASE DRIFT POLYNOMIAL (P)			
		2 (Linear frequency drift)	3 (quadratic frequency drift)	4 (cubic frequency drift)	P
ORDER OF THE PHASE INCREMENT (M)	1	time dependent	time dependent	time dependent	time dependent
	2	$\sigma_y(\tau) = \frac{d_1}{\sqrt{2}} \tau$	time dependent	time dependent	time dependent
	3	0	$\Sigma_y(\tau) = \frac{2}{3} d_2 \tau^2$	time dependent	time dependent
	4	0	0	$\sigma_{HBC} = 6 d_3 \tau^3$	time dependent
	M = P	0	0	0	$\sigma_{HBC} = (P-1)! d_{P-1} \tau^{P-1}$

random fluctuations in oscillators: however, nonrandom variations (drifts) of the output frequency do exist in most devices and can be sometimes modeled by deterministic polynomial functions [9]. In this context, structure functions are also of interest since they provide a means to obtain insight into the highest order drift coefficient, i.e., into the long-term instability of the source [50].

Let us assume that the oscillator frequency drift is modeled by a (P - 1)th degree polynomial (P ≥ 2):

$$y(t) = \sum_{i=1}^{P-1} d_i t^i \tag{8.12}$$

Integrating from 0 to t yields an Pth degree polynomial for the phase drift:

$$\varphi(t) = \sum_{i=1}^{P-1} D_i t^{i+1} \tag{8.13}$$

where the coefficients D_i are related to the d_i by

$$D_i = \frac{\omega_0 d_i}{i + 1} \tag{8.14}$$

A frequency measurement beginning at t_k and lasting τ seconds yields the average value (ignoring random fluctuations):

$$\bar{y}_k = \sum_{i=1}^{P-1} \frac{d_i}{(i + 1)\tau} [(t_k + \tau)^{i+1} - t_k^{i+1}]. \tag{8.15}$$

Substituting this expression into (8.8) or (8.9) allows one to study the properties of the successive phase increments in the presence of polynomial drifts:

- a) the phase increments of order M smaller than P are time dependent and hence are of little use as measures of the drift constant coefficients;

- b) the phase increments of order M greater than P are all equal to zero and hence are meaningless;
- c) only the P th phase increment ($M = P$) as time-independent and nonzero:

$$\Delta^{(P)}\varphi(t; \tau) = (P - 1)! \omega_0 d_{P-1} \tau^P. \tag{8.16}$$

It is directly related to the highest order drift coefficient d_{P-1} , and provides thus a measure of long-term instability since $d_{P-1} t^{P-1}$ becomes the dominant term in (8.12) as t increases [50]. Of course, real oscillators are equally perturbed by random noises and the higher phase increments will not be zero but rather will provide a measure of the noise through relation (8.10).

For a better comparison among drifts and random instabilities, it is of interest to calculate parameters such as $\sigma_y(\tau)$, $\Sigma_y(\tau)$ or others for polynomial drifts; for that, expression (8.16) may be substituted into the quantities appearing in Table VII. The results are shown in Table VIII: for a given value $P - 1$ of the degree of $y(t)$, only one parameter is of interest and varies as τ^{P-1} : e.g., $\sigma_y(\tau)$ varies as τ for linear frequency drift and $\Sigma_y(\tau)$ as τ^2 for quadratic frequency drift.

The above results have been used to make an algorithm to test for the highest order of frequency drift (and also for the presence of power-law noises) based upon observing the behavior of samples of structure functions [50].

D. Structure Function of Frequency

From the mathematical point of view, it is also possible to deal with the successive increments of $y(t)$ defined as [52]:

$$\Delta^{(M)}y(t; \tau) = \sum_{k=0}^M (-1)^k \binom{M}{k} y(t + (M - k)\tau). \tag{8.17}$$

However, this quantity involves the instantaneous values of $y(t)$ which are not observables and hence its practical utility is

limited. Nevertheless, calculations show that $D_y^{(M)}(\tau)$ has general properties very similar to those of $D_\varphi^{(M)}(\tau)$ which is related to observables through equations (8.8) and (8.9).

1) *Random Instabilities*: In this case, $D_y^{(M)}(\tau)$ is related to $S_y(f)$ by

$$D_y^{(M)}(\tau) = 2^{2M} \int_0^\infty S_y(f) \sin^{2M} \pi \tau f df \quad (8.18)$$

which is analog to (8.11) for phase noise. Again, higher values of M allow one to deal with more divergent power laws.

2) *Polynomial Drifts*: From the model given by equation (8.12), it can be demonstrated that:

- a) the frequency increments of orders M smaller than $P - 1$ are time-dependent and hence not very useful;
- b) the frequency increments of orders M greater than $P - 1$ are zero, and hence meaningless;
- c) only the $(P - 1)$ th frequency increment is both nonzero and time-independent and reads as:

$$\Delta^{(P-1)}y(t; \tau) = (P - 1)! d_{P-1} \tau^{P-1} \quad (8.19)$$

It is directly related to the highest order drift coefficient in $y(t)$, and is equal to σ_{HBC} appearing in Table VIII.

As particular cases, for $P = 2$ (linear frequency drift):

$$\Delta^{(1)}y(t; \tau) = \sqrt{2} \sigma_y(\tau) \quad (8.20)$$

and for $P = 3$ (quadratic frequency drift):

$$\Delta^{(2)}y(t; \tau) = 3 \sum_y(\tau). \quad (8.21)$$

Thus structure functions of frequency applied to polynomial drifts, provide essentially the same information as structure functions of phase.

E. Long-Term Instability of Oscillators

For great values of τ (say, days, months, etc), significant frequency instabilities may arise from both very slow random fluctuations and very slow deterministic frequency drifts; they are modeled, respectively, by power-law spectral densities with negative slopes and by polynomials. Structure functions allow one to deal with both kinds of processes as shown above (see (8.10) and (8.16)): it is then of interest to compare the laws obtained for the relevant parameters.

A problem is that the M th-order structure function of phase allows one to deal with several spectral densities $S_y(f) = h_\alpha f^\alpha$, namely down to $\alpha > -(2M - 1)$, whereas it yields a significant value only for the M th-order polynomial phase drift $\varphi(t) = D_{M-1} t^M$. As a consequence, with a unique statistical parameter (i.e., derived from a fixed order structure function) it is not possible to establish a full correspondence between power laws and polynomial drifts in the sense that one sample of each would yield a significant law for the parameter of interest.

The results presented in Table IX show the laws of $\sigma_y(\tau)$, $\Sigma_y(\tau)$ and $\sigma_{HBC}(N, \tau, \tau)$ for power laws with integral negative slopes and polynomial drifts: generally speaking, these parameters follow a positive-slope power law versus τ , with increasing slope when either the negative slope of $S_y(f)$ or the order of the polynomial drift increases.

More specifically, for random processes, the maximum positive slope versus τ of the standard-deviation derived from the M th-order structure function of phase is $M - 3/2$ for integral

TABLE IX
LONG-TERM INSTABILITY: SLOPES OF TIME-DOMAIN
PARAMETERS FOR RANDOM NOISES AND
POLYNOMIAL DRIFTS

$S_y(f)$	$\sigma_y(\tau)$	$\Sigma_y(\tau)$
$h_{-1} f^{-1}$	τ^0	τ^0
$h_{-2} f^{-2}$	$\tau^{1/2}$	$\tau^{1/2}$
$h_{-3} f^{-3}$	∞	τ
$h_{-4} f^{-4}$	∞	$\tau^{3/2}$
$h_{-5} f^{-5}$	∞	∞
max slope for integral values of α $M - \frac{3}{2}$	1/2	3/2

$y(t) = d_{P-1} t^{P-1}$	Parameters of interest Slope : $P - 1$
$d_1 t$	$\sigma_y(\tau) \sim \tau$
$d_2 t^2$	$\Sigma_y(\tau) \sim \tau^2$
$d_3 t^3$	$\sigma_{HBC} \sim \tau^3$
$d_4 t^4$	$\sigma_{HBC} \sim \tau^4$

values of α (e.g., $\sigma_y(\tau)$ derived from the second order phase structure function has a maximum positive slope of 1/2). For nonintegral values, the actual maximum is $< M - 1$. On the other hand, for a drift $y(t) = d_{P-1} t^{P-1}$, only one standard-deviation is of interest and follows the law τ^{P-1} .

Thus it is impossible to have a specific power law (α fixed) and a specific drift order yielding the same slope versus τ for the same time-domain parameter (of course, different parameters may follow the same law, e.g., $\Sigma_y(\tau) \sim \tau$ for $S_y(f) = h_{-3} f^{-3}$ and $\sigma_y(\tau) \sim \tau$ for $y(t) = d_1 t$).

As a consequence, a careful study of the laws of $\sigma_y(\tau)$, $\Sigma_y(\tau)$ and $\sigma_{HBC}(N, \tau, \tau)$ versus τ should allow one to determine unambiguously whether the frequency standard is disturbed by such or such random noise and/or by such or such polynomial drift; these conclusions are summarized in Table X. Note that these comments are true only for integral values of α in the power law spectra. If α can take on fractional powers, say $\alpha = -2.8$, then $\sigma_y(\tau) \sim \tau^{+0.9}$ and it is convergent. As α gets close to -3 but $\alpha > -3$, $\sigma_y(\tau) \rightarrow \tau^{+1}$ and remains convergent.

Of course, visual inspection of an historical graph of \bar{y}_k versus time may be very helpful.

Also, for finite data sets, a spectra of $h_{-3} f^{-3}$ or $h_{-5} f^{-5}$ may well give false diagnosis even though $\sigma_y(\tau)$ and $\Sigma_y(\tau)$ are not convergent for these noises.

TABLE X
IDENTIFICATION OF RANDOM AND DETERMINISTIC POLYNOMIAL VARIATIONS FROM THE LAWS OF $\sigma_y(\tau)$ AND $\Sigma_y(\tau)$

The law of $\sigma_y(\tau)$	indicates the presence of
$\tau^{1/2}$	random walk frequency noise
τ	linear frequency drift

The law of $\Sigma_y(\tau)$	indicates the presence of
$\tau^{1/2}$	random walk frequency noise
τ	$S_y(f) = h_{-3} f^{-3}$
$\tau^{3/2}$	$S_y(f) = h_{-4} f^{-4}$
τ^2	quadratic frequency drift

F. Summary

The structure function approach introduced by Lindsey and Chie appeared very fruitful in terms of:

- theoretical unification of the various parameters proposed as time domain measures;
- new possibilities for dealing with power-law spectral densities of random noises and with polynomial drifts.

In this paper, the unification process has been developed by showing the relations between $D_\varphi^{(M)}(\tau)$ and previous parameters for $M \geq 3$ (see Table VII).

Also, the use of structure functions for identifying power law type random noises when polynomial drifts are present has been outlined (Tables IX and X), but great care must be exercised since false diagnosis may arise.¹⁵

IX. FREQUENCY INSTABILITY MEASUREMENT

Even the most sophisticated statistical parameters are of no practical value for oscillator specification if they cannot be measured experimentally, i.e., estimated from finite samples of data. To this end, in parallel with the theoretical developments reviewed in this paper, much work has been devoted in the laboratories to the implementation of measurement test-sets capable of resolving the very small fluctuations of precision frequency sources. These systems rely heavily on the development of modern components and instrumentation such as low noise mixers/amplifiers and digital counters or spectrum analyzers.

It is not our intent to give here technical descriptions of these systems that have been described with great details in many references (see for example: [1], [2], [19], [21], [22], [38], [41], [44], [45], [54], [55], [57]). The goal is rather

¹⁵During the review process of this paper, reference [70] has been published where emphasis is put on the identification of power-law type spectral densities. Note that the parameter $D_x^{(3)}(\tau)$ appearing in their Table I is related to $\Sigma_y^2(\tau)$ in our Table V by $\Sigma_y^2(\tau) = D_x^{(3)}(\tau)/9\tau^2$.

to emphasize some of their features. Readers are referred to original references for circuit descriptions.

A. The Need for a Reference Oscillator

To measure the frequency of one oscillator, at least two oscillators are indeed needed, e.g., one of them providing the time base of the digital counter (this is of course a particular case of the need for a reference in any physical measurement).

As far as the reference oscillator has a better stability than the oscillator under test, measured instabilities are due mainly to the latter. But when dealing with state-of-the-art sources, both oscillators are often of the same type and have nearly the same quality; the assumption is then made that they have uncorrelated random frequency fluctuations with identical statistical properties: the measured standard deviations must then be divided by $\sqrt{2}$ to get the contribution of one oscillator. A method has been proposed for estimating frequency instability of an individual oscillator, using data obtained by comparing it with at least two other oscillators [53], assuming their FM noises are uncorrelated; the method works even if the "reference" oscillators are less stable than the oscillator under test, although confidence in the results deteriorates rapidly if the test oscillator is much better than the references.

B. Frequency Versus Time-Domain Measurements

The dichotomy that appeared in the theory is of course found again in measurement apparatus: classically, digital counters are used for time-domain measurements and low-frequency spectrum analyzers for frequency-domain measurements.

However, theoretical developments leading to parameters such as the Hadamard variance or the high-pass variance have attenuated this dichotomy, since spectral densities may be measured by counting techniques [31], [39], [40] and variances may be measured by filtering techniques [43], [46].

Generally speaking, frequency-domain and time-domain measurement systems have several common parts even when

the output measurement apparatus (e.g., a counter or a spectrum analyzer) is not the same.

C. The Need for Demodulation

Since we want to measure the phase or frequency fluctuations of the oscillator quasi-sinusoidal output signal, techniques for demodulating these fluctuations are required before measurement and statistical analysis.

A PLL wherein a reference oscillator is locked to the oscillator under test, the two signals being then in phase quadrature at the phase detector inputs, is often used for that purpose: the PLL time constant is chosen as to provide a phase detector output analog voltage proportional either to the phase or to the frequency fluctuations over the Fourier frequency range of interest [22], [44], [45], [59]. According to the settings and the measurement apparatus used, the PLL system may be used either for frequency or for time domain measurements.

Sometimes, dispersive elements are used as frequency discriminators to convert frequency noise into an analog voltage [54].

D. The Need for High-Performance Systems

The state-of-the-art of time and frequency standards has advanced to such a level that a single measurement apparatus is unable to resolve directly their very low fluctuations by lack of resolution, sensitivity or dynamic range.

Techniques are therefore needed to enhance the fluctuations before measurement: for example, frequency multipliers followed by heterodyning, or frequency error multipliers have been used for that purpose. For the same reason, a very low noise amplifier is included between the phase detector in the PLL technique and the measurement apparatus.

The system noise must also be kept at a lower level than the oscillator noise level to be measured; in this respect, the advent of Schottky barrier diode mixers (used as phase detectors) has been a significant breakthrough in implementing low-noise measurement test-sets. It is recommended to measure system noise before any measurement on oscillators as shown in [55].

E. Automated Measurements

Following a general trend, automated measurement systems have been developed for both frequency and time domains [38], [54], [56], [58]. Automated systems are really useful in this field since great numbers of data have to be processed, various statistical treatments are made yielding different kinds of parameters and graphical plots of results are often needed. For these reasons, a calculator program may include frequency selection, bandwidth settings, measurement sequence, statistical treatments, results plotting and even system calibration.

F. A Hierarchy of Measurement Systems

While developing a high-resolution time difference measurement system, Allan and Daams [59] have set up the following hierarchy of measurement systems according to the quantity that they can measure:

- a) time;
- b) time fluctuations;
- c) frequency;
- d) frequency fluctuations.

For a system capable of measuring say time fluctuations, frequency and frequency fluctuations can be deduced from the

results, but time cannot; and so on for the other systems in this hierarchy. The systems of status a) with state-of-the-art performance provide the greatest flexibility in data processing, i.e., systems where time differences can be measured with adequate precision. However, a system of status (d) is quite satisfactory when frequency instability is the primary concern: this is the case of the tight PLL system where the time constant is of the order of a few milliseconds: the output voltage is then proportional to the frequency (and not to the phase) fluctuations for averaging times longer than the loop time constant. This system is useful for time-domain measurements with τ of the order of one second and longer.

X. CONCLUSIONS

We have reviewed several proposed phase and frequency stability measures including both well-known widely used parameters and new concepts. For the latter, we have tried to point out clearly their specific advantages through the study of their transfer functions. In connection with structure functions, the effects of polynomial drifts of great importance for long term have been outlined and compared with those of LF divergent random noises.

In all the approaches, spectral densities play a key role in the sense that the other measures may be deduced from them whereas general inversion of the formulas is usually very difficult if not impossible.

In February 1978, Study Group 7 on "Standard frequencies and time signals" of the International Radio Consultative Committee (CCIR) has adopted a new document (to be included in [62]) recommending the use of spectral densities and, for time domain, the use of the two-sample standard deviation $\sigma_y(\tau)$. This new recommendation is certainly of great importance at the international level for nonspecialist engineers faced with the problem of stability characterization.

Nevertheless, it is felt that experimenters should be encouraged to try some of the new parameters since they may prove useful, especially for research work where better insight into the oscillator fundamental properties is looked for.

At this point, it must be emphasized that other analysis techniques not covered in this paper have been developed with possible applicability to frequency sources specification. Especially, the fitting of Auto-Regressive Integrated Moving Average (ARIMA) models to the data provides a powerful means of computer simulation and future values optimal prediction [63]. Barnes has discussed their applicability to time and frequency data, including the relation between the model parameters and $S_y(f)$ [12]. With more details, ARIMA models have been used by Percival [64] for predictions of fractional frequency of commercial cesium beam standards with lead times from 1 to 64 days, and by Hübner [65] for prediction in the realization of time scales.

Of special interest in connection with frequency stability characterization, Percival has proposed to use prediction errors as a measure of frequency instability, i.e., frequency instability is associated with a measure of frequency unpredictability which automatically incorporates the systematic terms.

Other new approaches have been recently proposed such as the finite-time variance defined as the mean of a general quadratic form of the measurements [66], and the suggested applicability of ambiguity functions for stability characterization when the oscillator is included in a global system whose performance are looked for [67].

At present, the existing models for random gaussian noises are well documented and provide a good background for actual measurements interpretation. However, future researches will certainly include the development of more sophisticated models that may possibly improve our understanding of some oscillator fundamental properties, e.g., of flicker noises and possible sporadic elements. These new models will probably call for new stability measures.

Thus although it appears to be mature, the field is still widely open for research. As in the past, no doubt that the development of complex systems, such as digital communications [68], with stringent requirements as regards frequency sources and clocks specification will provide a strong motivation.

GLOSSARY OF SYMBOLS

Many symbols are used throughout this paper. The most important ones are defined below. For clarity, they are presented under several general headings.

Symbols	Definitions	See Equation
	Signal	
$V(t)$	Instantaneous output voltage of a (quasi) sinusoidal signal generator.	(2.1); (2.2); (2.4)
V_0	Nominal amplitude of $V(t)$.	(2.1); (2.2); (2.4)
ν_0	Nominal frequency of $V(t)$ (angular frequency $\omega_0 = 2\pi\nu_0$).	(2.1); (2.2); (2.4)
	Fluctuations, Modulations, and Drifts	
$\epsilon(t)$	Random instantaneous fluctuations of amplitude (amplitude noise).	(2.2)
$\varphi(t)$	Random instantaneous fluctuations of phase about $2\pi\nu_0 t$ (phase noise).	(2.3); (2.4)
$\Delta\nu(t)$	Random instantaneous fluctuations of frequency about ν_0 (frequency noise).	(2.6); (2.7)
$\nu(t)$	Instantaneous frequency of $V(t)$.	(2.6)
$x(t)$	Random instantaneous time error.	(2.10)
$y(t)$	Fractional fluctuations of frequency.	(2.9)
$A(t)$	Fractional fluctuations of amplitude.	(3.13)
f_m	Modulating frequency for a sinusoidal FM or φM .	(2.3); (5.14)
d_i	Polynomial drift coefficient in fractional frequency (i th order).	(5.17); (8.12)
D_i	Polynomial drift coefficient in phase ($i + 1$)th order.	(2.3); (8.13)

	Frequency Domain	
$S_g^{(TS)}(f)$	Two-sided spectral density on a per hertz basis of the real function $g(t)$.	(3.2)
$S_g(f)$	One-sided spectral density of $g(t)$. Applied in the paper to: $\varphi(t)$, $\Delta\nu(t)$, $x(t)$, $y(t)$, $A(t)$, $V(t)$.	(3.3)
f	Fourier frequency ($\omega = 2\pi f$).	
α	Exponent of f for a power-law spectral density (usually: $\alpha = 2, 1, 0, -1, -2$).	(3.9)
h_α	Positive real coefficient of f^α in a power-law spectral density.	(3.9)
f_h	HF cutoff of an ideal sharp cutoff low-pass filter.	(3.10)
	Measurement Sequence	
τ	Averaging time interval for frequency measurements.	(4.1)
t_k	The instant of the beginning of the k th measurement of frequency ($k = 1, 2, 3, \dots, N$).	(4.1)
T	Time interval between the beginnings of two successive measurements of frequency ($T = t_{k+1} - t_k$).	Fig. 4
N	Number of data for calculating a sample variance, a modified sample variance or a Hadamard variance (positive integer).	Fig. 4
m	For $N = 2, m + 1$ data are used to estimate $\sigma_y^2(\tau)$.	(4.22)
$h(t)$	A real function of time that resembles the measurement sequence of a time-domain parameter. Applied to several variances in the text.	(5.3)
	Time Domain	
y_k	Instantaneous value of $y(t)$ at $t = t_k$.	(7.9)
\bar{y}_k	Fractional frequency fluctuations averaged from t_k to $t_k + \tau$.	(4.2)
n_k	A counting result: number of input signal cycles between t_k and $t_k + \tau$.	(4.1)
$I^2(\tau)$	The true variance of \bar{y}_k .	(4.5)
$\sigma_y^2(N, T, \tau)$	Sample variance of N samples \bar{y}_k ($k = 1$ to N). Three definitions are given in the text. This symbol applies to the second one.	(4.13)

$\sigma_y^2(\tau)$	The above for $N = 2$ and $T = \tau$. A recommended measure of instability in the time domain.	(4.17);(4.18)	P	The order of a polynomial drift.	(8.12)
$\hat{\sigma}_y^2(\tau, m)$	An estimate of $\sigma_y^2(\tau)$ obtained from $m + 1$ measurements of \bar{y}_k .	(4.22)		ACKNOWLEDGMENT	
E_α	Confidence interval for the above estimate and a power law spectral density $h_\alpha f^\alpha$.	(4.23)		The author would like to express his thanks to numerous people with whom he had very fruitful discussions during the last few years, in particular to D. W. Allan, Dr. J. A. Barnes, Prof. J. de Prins and Prof. B. Picinbono. Also, the constructive discussions with Dr. A. R. Chi, Dr. P. Kartaschoff, and Dr. G. M. R. Winkler during the last two meetings of Study Group 7 of the CCIR were greatly appreciated. Lastly, special thanks are due to Prof. J. Uebersfeld for his constant interest and encouragement.	
$\langle \sigma_H^2(N, T, \tau) \rangle$	The Hadamard variance with N samples $\bar{y}_k (k = 1 \text{ to } N)$.	(6.1)		REFERENCES	
$\langle \sigma_{HBC}^2(N, T, \tau) \rangle$	The Hadamard variance where the samples are weighted by binomial coefficients.	(6.4)		[1]	"IEEE-NASA Symposium on Short-Term Frequency Stability," Goddard Space Flight Center, Greenbelt, MD, Nov. 23-24, 1964 (Document NASA SP-80).
$\sigma_{HP}^2(\tau)$	The high-pass variance (measured by filtering).	(6.8);(6.10)		[2]	Special Issue on Frequency Stability, <i>Proc. IEEE</i> , vol. 54, pp. 101-338, Feb. 1966.
$\sigma_{BP}^2(\tau)$	The bandpass variance (measured by filtering).	(6.8); Table IV		[3]	E. J. Baghdady, R. N. Lincoln, and B. D. Nelin, "Short term frequency stability: Characterization, theory and measurement," in [1], pp. 65-88.
$\Sigma_y^2(N, T, \tau)$	A modified sample variance of N samples $\bar{y}_k (k = 1 \text{ to } N)$.	(7.1)		[4]	L. S. Cutler, "Some aspects of the theory and measurement of frequency fluctuations in frequency standards," in [1], pp. 89-100.
$\Sigma_y^3(\tau)$	The above for $N = 3$ and $T = \tau$.	(7.4)		[5]	L. S. Cutler and C. L. Searle, "Some aspects of the theory and measurement of frequency fluctuations in frequency standards," in [2], pp. 136-154.
$\langle \delta y_T^2 \rangle$	The mean square of the fractional frequency increment $\delta y_T = y(t_{k+1}) - y(t_k)$ where $T = t_{k+1} - t_k$.	(7.9)		[6]	R. Vessot, L. Mueller, and J. Vanier, "The specification of oscillator characteristics from measurements made in the frequency domain," in [2], pp. 199-207.
$\langle \delta \bar{y}_T^2 \rangle$	The mean square of the difference between two successive non adjacent $\bar{y}_k (T > \tau)$.	(7.15)		[7]	D. W. Allan, "Statistics of atomic frequency standards," in [2], pp. 221-230.
$R_g(\tau)$	Correlation function of the real function $g(t)$.	(3.1)		[8]	J. A. Barnes, A. R. Chi, L. S. Cutler, D. J. Healey, D. B. Leeson, T. E. McGunigal, J. A. Mullen, W. L. Smith, R. L. Sydnor, R. Vessot and G. M. R. Winkler, "Characterization of frequency stability," <i>IEEE Trans. Instrum. Meas.</i> , vol. IM-20, pp. 105-120, May 1971.
	Transfer Functions			[9]	G. M. R. Winkler, "A brief review of frequency stability measures," in <i>Proc. 8th Annu. PTTI Applications and Planning Meeting</i> , (U.S. Naval Research Laboratory, Washington DC), Nov. 30-Dec. 2, 1976, pp. 489-527.
$H(f)$	The transfer function associated with a variance and appearing in its relationship with $S_y(f)$. It is the Fourier Transform of $h(t)$. Applied to several variances in the text.	(3.1);(6.7)		[10]	J. A. Barnes, "A review of methods of analyzing frequency stability," in <i>Proc. 9th Annu. PTTI Applications and Planning Meeting</i> , Nov. 29-Dec. 1, 1977.
$H_\varphi(f)$	The transfer function appearing in the relationship between a variance and $S_\varphi(f)$.	(6.8);(6.9);(6.10)		[11]	D. Slepian, "On bandwidth," <i>Proc. IEEE</i> , vol. 64, pp. 292-300, Mar. 1976.
	Structure Functions			[12]	J. A. Barnes, "Models for the interpretation of frequency stability measurements," NBS Tech. Note 683, Aug. 1976.
M	The order of an increment or of a structure function (positive integer).			[13]	E. Boileau and B. Picinbono, "Statistical study of phase fluctuations and oscillator stability," <i>IEEE Trans. Instrum. Meas.</i> , vol. IM-25, no. 1, pp. 66-75, March 1976.
$\Delta^{(M)}g(t; \tau)$	The M th increment of the real function $g(t)$.	(8.1)		[14]	J. L. Doob, "The Brownian movement and stochastic equations," <i>Ann. Math.</i> , vol. 43, no. 2, Apr. 1942.
$D_g^{(M)}(T; \tau)$	The structure function of the M th increment.	(8.3)		[15]	A. M. Yaglom, "Correlation theory of processes with random stationary N th increments," <i>Ann. Math. Soc. Trans.</i> , vol. 8, pp. 87-141, 1958.
$D_g^{(M)}(\tau)$	The M th structure function of $g(t)$ defined as $D_g^{(M)}(T = 0; \tau)$.	(8.4)		[16]	B. Picinbono, "Processus à accroissements stationnaires," <i>Ann. des Télécom.</i> , Tome 30, no. 7-8, pp. 211-212, July-Aug. 1975.
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